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Optimal liquidation using extended trading close for multiple trading days

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Abstract

The extended trading close (ETC) provides institutional investors an opportunity to trade at the closing price after the regular trading session (RTS) and disclosing the order imbalances to other market participants. ETCs exist in the Nasdaq, the SSE STAR, the SZSE ChiNext and the TWSE. To help a risk-averse institutional investor take advantage of the RTS and the ETC for liquidation, we develop a multistage dynamic programming model including the ETC, and derive recursive solutions for the multiple trading days scenario with closed-form solutions for the scenario with only two trading days. We also verify that the ETC is able to mitigate extreme price movements caused by fast liquidation, which is also a goal of the ETC set out by the SSE STAR and the SZSE ChiNext. Finally, we derive three results. First, an institutional investor can reduce execution costs after the introduction of the ETC. Second, a critical trading day exists, and to avoid prematurely revealing trading intentions, the investor should not trade in the ETC until such day. Third, even though the ETC orders submitted by the investor are unfilled, implementation of the ETC encourages the investor to change the liquidation strategy in the RTS, which reduces extreme price movements. In summary, the practical implications of this paper are that the investor should not trade during the ETC on the front few days to avoid prematurely revealing the investor's trading intention by unfilled orders in the ETC and that introducing the ETC can reduce liquidation costs and extreme price movements.

Keywords: Extended trading close, Optimal liquidation, Market impact, Market microstructure

JEL Classifcation: C61, G11, G18

Introduction

Background and motivation

We consider an institutional investor who uses a regular trading session (RTS) and the extended trading close (ETC) to liquidate a large block over multiple trading days. At the end of each trading day's RTS, the investor has one opportunity to submit an order in the ETC. The investor needs to not only decide on which trading days to submit orders in the ETC and the volume of orders, but also adjust the liquidation strategy in the RTS owing to the introduction of the ETC. We obtain optimal liquidation strategies using the

ETC. In addition, the ETC can save investors execution costs and reduce extreme price movements[1] caused by the fast liquidation.

When an investor liquidates in the RTS, they consume a large amount of liquidity. This action results in liquidity price impacts so that the investor bears the liquidity costs (Tsoukalas et al. 2019). To help institutional investors reduce the liquidity costs, many exchanges[2] have introduced ETCs; for example, the Nasdaq introduced the ETC on March 7, 2022. The Nasdaq (2021)[3] describes the ETC as a trading system that fixes the price at the closing price[4] after market close and discloses order imbalances (the sizes of unfilled orders). We present the trading rules of ETCs for different exchanges in Appendix 6.

Since the price of the ETC is fixed at the closing price and the trading hours do not coincide with the RTS, investors trading in the ETC do not suffer from liquidity price impacts and do not miss the opportunity to trade in the RTS. Although the ETC has the abovementioned advantages, it produces two new problems. The first is information price impacts (Zhu et al. 2023); that is, unfilled and exposed orders in the ETC affect the price in the future. Specifically, when other market participants observe unfilled orders, they suspect that there is private information in these orders and change their trading strategy so that it affects the price in the future. The second is execution uncertainty, which means that not all orders in the ETC are executed. Specifically, the investor cannot change the price to gain more liquidity to ensure that the orders are executed. In conclusion, the investor is not subject to the liquidity costs in the ETC, although they can suffer from information price impacts and execution uncertainty.

We consider a risk-averse investor who has to liquidate a fixed amount of an asset within N trading days, where N is an integer greater than zero. On each trading day, the investor first trades the asset in the RTS and then in the ETC. In the RTS, orders submitted by the investor are guaranteed to be executed but have liquidity price impacts on the current prices. In the ETC, such orders submitted do not create an impact on the current prices, but if not executed, they create information price impacts on the future prices. The investor needs to trade off liquidity price impacts against information price impacts and allocate the orders between the RTS and the ETC. We derive recursive solutions, and the solutions exist on a critical trading day. Before reaching the critical trading day, the investor should not submit orders in the ETC. Thereafter (except for the last trading day), the investor should submit orders in the ETC.

As a special case, we obtain closed-form solutions for two trading days ($N = 2$) and analyze some properties of the optimal liquidation strategy. First, we investigate the impact of introducing the ETC on the investor's trading strategy in the RTS and find that they slow trading in the RTS on the first trading day to wait for the ETC trading

[1] Brogaard et al. (2018) empirically employed extreme returns to represent extreme price movements. In this paper, extreme price movements are clearly caused by liquidity price impacts resulting from rapid liquidation. Therefore, we use liquidity price impacts to denote extreme price movements.

[2] For example, ETCs exist in the Nasdaq, the Science and Technology Innovation Board of Shanghai Stock Exchange (SSE STAR), the ChiNext Market of Shenzhen Stock Exchange (SZSE ChiNext) and the Taiwan Stock Exchange (TWSE).

[3] The Nasdaq proposed to the Securities and Exchange Commission to establish the ETC and a new ETC Order Type. Source: https://listingcenter.nasdaq.com/assets/rulebook/nasdaq/filings/SR-NASDAQ-2021-040.pdf.

[4] In the ETC, all market participants can submit orders only at the closing price. Therefore, the ETC gathers all the liquidity at this price.

opportunity. We then examine the factors influencing the optimal proportion of the submitted order volume in the ETC to the inventory and establish that the optimal proportion is positively correlated with the liquidity price impact coefficient, the asset volatility, and the investor's risk aversion coefficient.

In addition, we analyze the effect of introducing the ETC on the execution costs and extreme price movements, whereby we verify that the ETC can achieve the intended goals set out by the Nasdaq, the SSE STAR, the SZSE ChiNext and the TWSE[5]. Their goals are as follows. First, institutional investors' trading demands must be satisfied. Second, the price impact of block trades in the RTS must be reduced. Third, liquidity management methods[6] must be enhanced. Regarding the first goal, we compare the execution costs with and without the ETC and determine that the ETC can reduce the execution costs. Regarding the second goal, we compare the extreme price movements caused by fast liquidation with and without the ETC and discover that the ETC can reduce extreme price movements. The following two reasons explain this: the ETC can decrease the sizes of orders in the RTS, and the ETC encourages the investor to reduce their extreme trading speed in the RTS. Concerning the third goal, we demonstrate that the supervisor can provide liquidity[7] in the ETC to reduce extreme price movements. The second and third goals of introducing the ETC are important. Both the U.S.' and China's equity markets experienced extreme price movements caused by liquidity, such as the stock market crash in China in 2015 to 2016 and the extreme price movements in the US stock market on 6 May 2010 (Duffie and Zhu 2017).

The contribution of this paper is threefold. First, we derive an analytical solution to the optimal execution strategy for multiple trading days using ETCs, which enriches the literature on the optimal execution strategy. Second, we reveal that introducing the ETC reduces extreme price movements and the value function, which provides theoretical evidence for introducing ETCs in the SSE STAR and SZSE ChiNext. Finally, we demonstrate that both the value function and the extreme price movement decrease as the number of trading days increases.

Related literature

Our paper is related to two strands of literature in the field of optimal liquidation strategy (also called optimal execution strategy). The purpose of optimal liquidation strategies is to help investors liquidate within a limited period of time and to minimize the liquidation cost, which generally consists of the cost of price impacts and market risk.

The first strand is the optimal liquidation strategy, which originates from Bertsimas and Lo (1998) and Almgren and Chriss (2001), in the RTS. Bertsimas and Lo (1998) and Almgren and Chriss (2001) investigated the optimal liquidation strategies of risk-neutral

[5] Source: https://www.institutionalinvestor.com/article/b1wq64ps09pnmg/Will-Nasdaq-s-New-Extended-Trading-Close-ETC-Transform-After-Hours-Trading (for the Nasdaq); http://www.sse.com.cn/aboutus/mediacenter/hotandd/c/c_2019012 9_4717061.shtml (for the SSE STAR in Chinese); http://www.szse.cn/aboutus/trends/conference/t20200427_576498.html (for the SZSE ChiNext in Chinese).

[6] Specifically, the goal of introducing the ETC by the Nasdaq and TWSE is the first point. The goals of introducing the ETC by SSE STAR are the first and second points, and those of introducing the ETC by SZSE ChiNext are the first and third points.

[7] As an example where the supervisor provides liquidity, market turmoil occurred in China's stock market in the summer of 2015, and the Chinese government organized a "national team" of securities firms to backstop the market meltdown (Brunnermeier et al. 2020; Huang et al. 2019; Amstad et al. 2020).

and risk-averse investors, respectively. In addition, Schied and Schöneborn (2009) and Schied et al. (2010) transferred Almgren and Chriss (2001)'s assumption for the mean-variance utility function into absolute risk aversion (ARA) and constant absolute risk aversion (CARA), respectively. In addition, some studies have derived the optimal liquidation strategy for different features of the order book. For instance, Alfonsi et al. (2010) made a general assumption about the shape of the order book, and Bayraktar and Ludkovsk (2011) assumed that it obeys a Poisson process. Obizhaeva and Wang (2013) and Tsoukalas et al. (2019) assumed that the order book would experience exponential recovery and then derived the optimal liquidation strategy for a single asset and a portfolio, respectively. In contrast, in our model, investors can liquidate not only in the RTS but also the ETC.

Second, our paper relates to the literature on the optimal liquidation strategy, including the dark pool. The dark pool shares the same property with the ETC, that is, fixed price. The differences between the dark pool and ETC are as follows. First, unfilled orders in the dark pool are not exposed, which means that these do not generate information price impacts, whereas those in the ETC are exposed. Second, most dark pool trading times coincide with the RTS, whereas the trading hours of the RTS and the ETC do not overlap. Kratz and Schöneborn (2014) derived the optimal liquidation strategy involving the dark pool and the RTS. Kratz and Schöneborn (2015) obtained the optimal liquidation strategy for continuous-time on the basis of Kratz and Schöneborn (2014). Kratz and Schöneborn (2018) considered the adverse selection of the dark pool and obtained an analytical expression. Horst and Naujokat (2014) considered not only the adverse selection of the dark pool but also the elasticity of the liquidity. Owing to the aforementioned differences between the ETC and the dark pool, their conclusions are quite different from ours. Kratz and Schöneborn (2014), Horst and Naujokat (2014) and Kratz and Schöneborn (2015) recommended that investors submit their entire inventory to the dark pool, and for the liquidation strategy, Kratz and Schöneborn (2018) recommended that submitting orders in the dark pool for large inventory. However, to avoid premature exposure of trading intentions, investors should submit orders in the ETC in the latter trading days and the order size should be less than or equal to the inventory.

Finally, Yang et al. (2023) derived an optimal liquidation strategy when the investor submits an ETC order in the first stage and only trades in the RTS afterwards. We extend Yang et al. (2023) to multiple trading days, which means that the investor needs to consider which trading days to submit ETC orders. In particular, the ETC motivates the investor to change their liquidation strategy in the RTS, whereas in Yang et al. (2023) the trading strategy in the RTS does not alter, regardless of whether the ETC exists. We further investigate the impact of the institutional investor's liquidation on extreme price movements through the investor's different trading strategies in the RTS with and without the ETC.

The remainder of this paper is organized as follows. In "The main model" section, we introduce the market model for the RTS and the ETC. In "Main results" section, we derive recursive solutions for the optimal liquidation strategy over multiple trading days and analyze the effect that introducing the ETC has on execution costs and extreme price movements. In "Closed-form solutions for two trading days" section, we derive closed-form solutions for two trading days. In "The model with permanent price impact"

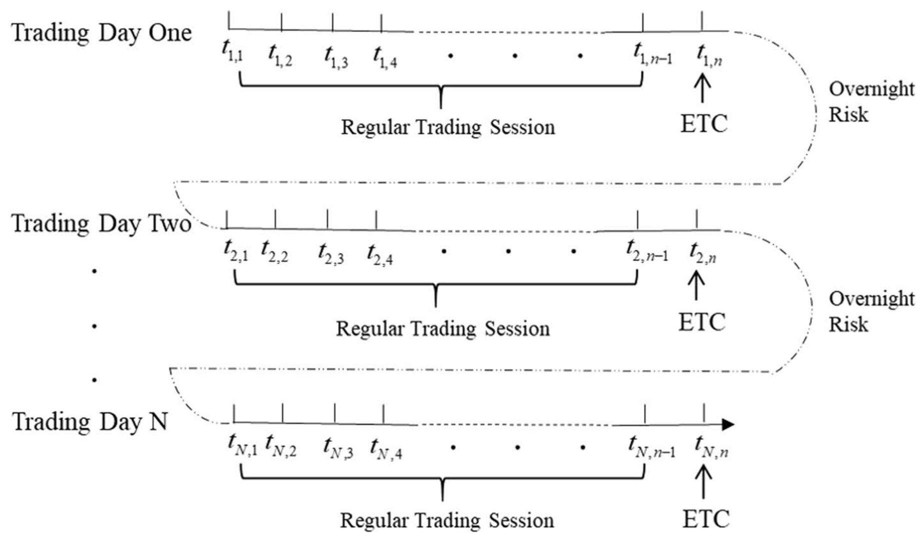

Fig. 1 Timeline

section, we derive recursive solutions for the optimal liquidation strategy with permanent price impact. Finally, we present concluding remarks in "Conclusion and future research" section.

The main model

We consider an investor who has to liquidate $X_{1,1}$ units of an asset within N trading days, where $X_{1,1} > 0$. They have n stages on each trading day spread across the RTS and the ETC. We denote the stage j on trading day i by $t_{i,j}$,[8] where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n$. On trading day i, we assign $\{t_{i,1}, t_{i,2}, \ldots, t_{i,n-1}\}$ to the RTS and $t_{i,n}$ to the ETC. We let $X_{i,j}$ denote the investor's inventory at $t_{i,j}$. At the end of trading day N, the investor liquidates the inventory completely, that is, $X_{N+1,1} = 0$. The timeline of this model is presented in Fig. 1.

The RTS and the ETC

Within trading day i, the investor submits a market order of $z_{i,j}$ units in the RTS at $t_{i,j} \in \{t_{i,1}, t_{i,2}, \ldots, t_{i,n-1}\}$ and an order of $z_{i,n}$ units in the ETC at $t_{i,n}$, where $z_{i,j} > 0$ and $z_{i,n} > 0$ mean "selling." In the RTS, the investor can procure sufficient liquidity from the variation in the price so that the order is executed, which means that they can always reduce the price to sell more, so the size of submitted order $z_{i,j}$ equals the filled order $a_{i,j}$. However, all participants can only trade at the closing price in the ETC so that the investor could not obtain sufficiently liquidity through reduced prices; thus, the investor's orders are not always executed in the ETC. Let $a_{i,n}$ denote the size of the order executed on trading day i in the ETC:

$$a_{i,n} = \begin{cases} z_{i,n}, & p, \\ 0, & 1 - p, \end{cases} \tag{1}$$

[8] We follow Kratz and Schöneborn (2014) in setting that the time between each stage is not necessarily equal, but rather adjusted for the U-shaped diurnal pattern of intraday volatility.

where $i = 1, 2, \ldots, N$. We adopt the assumption on the dark pool as in Horst and Naujokat (2014) and Kratz and Schöneborn (2014, 2015, 2018). We suppose that the ETC order is executed fully or not at all. We denote the probability of a successful execution by p.

The investor's inventory at the first stage of trading day i is

$$X_{i,1} = X_{i-1,n} - a_{i-1,n}, \qquad i = 2, 3, \ldots, N. \tag{2}$$

where $X_{1,1}$ is exogenous as given. The inventory at $t_{i,j}$ is

$$X_{i,j} = X_{i,j-1} - a_{i,j-1}, \tag{3}$$

where $i = 1, 2, \ldots, N, \quad j = 2, 3, \ldots, n$. In the RTS, the investor's transactions have a temporary impact on the price of the asset. The price of the asset in the RTS at $t_{i,j}$ is

$$P_{i,j} = \tilde{P}_{i,j} - \Lambda_1 z_{i,j} - \Lambda_2 \sum_{q=1}^{i-1} \left(z_{q,n} - a_{q,n} \right), \tag{4}$$

where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n-1$. The first term $\tilde{P}_{i,j}$ of (4) is the fundamental price of the asset. The second term $\Lambda_1 z_{i,j}$ of (4) is the "liquidity price impact," where $\Lambda_1 > 0$ is the coefficient of the liquidity price impact.[9] The liquidity price impact is generated when the investor consumes the liquidity on making transactions in the RTS. Specifically, instead of being able to complete all trades at the best price, the investor will trade along the limit order book. We present the relationship between price impact coefficient and the order book in detail in Appendix 1. For convenience of computation, we assume that the liquidity price impact is linear for the sizes of orders $z_{i,j}$ following Almgren and Chriss (2001), Kratz and Schöneborn (2018) and Tsoukalas et al. (2019), which implicitly assumes that the block order book. The third term of $\Lambda_2 \sum_{q=1}^{i-1} \left(z_{q,n} - a_{q,n} \right)$ of (4) is the "information price impact," where $\Lambda_2 > 0$ is the coefficient of the information price impact, and $\sum_{q=1}^{i-1} \left(z_{q,n} - a_{q,n} \right)$ is the unfilled accumulated sizes of orders in the ETC before trading day i. Information price impact is due to the fact that unexecuted ETC orders prematurely expose the investor's trading intentions. For convenience, we denote $b_i := \sum_{q=1}^{i-1} \left(z_{q,n} - a_{q,n} \right)$, which represents the investor's cumulative unexecuted ETC order volume.

The empirical basis for information price impacts is as follows. Based on data from the TWSE, Zhu et al. (2023) found that unfilled ETC orders have a permanent impact on future prices. Similarly, limit orders are also exposed and unfilled. Both Hautsch and Huang (2012) and Brogaard et al. (2019) provided empirical evidence that unfilled limit orders have a permanent price impact.

The investor trades are fixed at the closing price in the ETC, which thus does not create a liquidity price impact. However, previous unfilled orders in the ETC generate an information price impact on the current closing price. Therefore, the asset price in the ETC at $t_{i,n}$ is

[9] For convenience, we assume that Λ_1 is constant, but in practice intraday liquidity varies regularly. The variation can be captured by the variation of the liquidity price impact coefficient Λ_1 with the number of intraday stages j. We can still obtain an optimal strategy similar to Proposition 3.1 through backward induction.

$$P_{i,n} = \tilde{P}_{i,n} - \Lambda_2 \sum_{q=1}^{i-1} \left(z_{q,n} - a_{q,n}\right), \qquad i = 1, 2, \ldots, N. \tag{5}$$

During trading day i, the dynamic process of the fundamental asset price is

$$\tilde{P}_{i,j+1} = \tilde{P}_{i,j} + \varepsilon_{i,j+1}, \tag{6}$$

where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n-1$. $\varepsilon_{i,j+1}$ denotes the market risk in the RTS, that is, the random price fluctuation in the RTS. $\varepsilon_{i,j}$ is a random variable with a mean value of 0 and standard deviation $\sigma_{i,j}$, and lacks linear correlations. The volatility rate in the RTS is constant, namely, $\sigma_{i,j} = \sigma$. The dynamic process of the overnight asset fundamental price is

$$\tilde{P}_{i+1,1} = \tilde{P}_{i,n} + \varepsilon_{i+1,1}, \tag{7}$$

where $i = 1, 2, \ldots, N-1$. $\varepsilon_{i+1,1}$ denotes the overnight risk and lacks linear correlation with $\varepsilon_{i,j}$, with a mean value of 0 and a standard deviation of $\sigma_{i+1,1} = \tilde{\sigma}$, where $\tilde{\sigma} \neq \sigma$ (in reality, typically $\tilde{\sigma} > \sigma$).

Liquidation problem

The set of investor decision variables after $t_{i,j}$ is

$$\vec{z}_{i,j} := \{z_{q,l} | q \cdot n + l \geq i \cdot n + j, \quad q = 1, 2, \ldots, N, \quad l = 1, 2, \ldots, n.\}. \tag{8}$$

There are some restrictions on the investor's decision variables. First, short-selling is not allowed, that is, $z_{q,l} \leq X_{q,l}$. Moreover, because of laws,[10] spoofing (submitting spurious orders to mislead other market participants and enacting false strategies) is prohibited, the orders submitted by the investor in the ETC should concur with the trading direction. We consider an investor who needs to liquidate, so that investor can only submit sell ETC orders, that is, $z_{q,n} \geq 0$. If the investor's trading strategy meets all specifications, that is, $z_{q,l} \in \vec{z}_{i,j}$ satisfies $z_{q,l} \leq X_{q,l}$, and $z_{q,n} \geq 0$ such that $X_{N+1,1} = 0$, then we call $\vec{z}_{i,j} \in \mathbb{A}_{i,j}$ the set of feasible trading strategies.

The investor's information set at stage $t_{i,j}$ consists of all information, including stages prior to $t_{i,j}$ (including $t_{i,j}$). Specifically,

$$\mathcal{F}_{i,j} = \{X_{q,l}, \tilde{P}_{q,l}, b_q | q \cdot n + l \leq i \cdot n + j\}. \tag{9}$$

We adopt the definition from Perold (1988) that the investor's implementation shortfall at $t_{i,j}$ is

$$R_{i,j} := \tilde{P}_{i,j} X_{i,j} - \sum_{q=i+1}^{N} \sum_{l=1}^{n} P_{q,l} a_{q,l} - \sum_{l=j}^{n} P_{i,l} a_{i,l}, \tag{10}$$

[10] For example, article 1.5 of "The judicial interpretations given by the Supreme People's Court and the Supreme People's Procuratorate on some problems about the handling of securities and futures market manipulation criminal cases" was introduced on 1 July, 2019.

which denotes the difference between the book value $\tilde{P}_{i,j}X_{i,j}$ and the actual liquidation value $\sum_{q=i+1}^{N}\sum_{l=1}^{n}P_{q,l}a_{q,l}+\sum_{l=j}^{n}P_{i,l}a_{i,l}$. The investor's execution cost $J_{i,j}$ at $t_{i,j}$ comprises the expectation of implementation shortfall and market risk:

$$J_{i,j}:=\mathbb{E}_{i,j}[R_{i,j}]+\alpha\mathbb{E}_{i,j}\left[\sum_{q=i+1}^{N}\sum_{l=1}^{n}\sigma_{q,l}^{2}X_{q,l}^{2}+\sum_{l=j+1}^{n}\sigma_{i,l}^{2}X_{i,l}^{2}\right], \tag{11}$$

where $\mathbb{E}_{i,j}[\bullet]:=\mathbb{E}[\bullet|\mathcal{F}_{i,j}]$ is the conditional expectation given the information set at $t_{i,j}$. Following Kratz and Schöneborn (2014, 2015, 2018), the market risk is the multiplication of the risk aversion coefficient, the square of the inventory and the variance of random fluctuations, that is, $\alpha\mathbb{E}_{i,j}\left[\sum_{q=i+1}^{N}\sum_{l=1}^{n}\sigma_{q,l}^{2}X_{q,l}^{2}+\sum_{l=j+1}^{n}\sigma_{i,l}^{2}X_{i,l}^{2}\right]$, where $\alpha>0$ is the risk aversion coefficient. The investor's value function $v_{i,j}\left(X_{i,j},b_i\right)$ at $t_{i,j}$ is

$$v_{i,j}\left(X_{i,j},b_i\right):=\inf_{\bar{z}_{i,j}\in\mathbb{A}_{i,j}}J_{i,j}. \tag{12}$$

The value function is a function of the inventory and cumulative unfilled orders in the ETC, indicating the minimized execution cost.

Main results

Optimal liquidation strategy

We can derive the recursive solutions for multiple trading days using backward induction. The solutions are summarized by the following proposition.

Proposition 3.1 *The investor's optimal sizes of submitted orders in the RTS are*

$$z_{i,j}^{*}=\begin{cases}\frac{\alpha\sigma^{2}+B_{i,j+1}}{\alpha\sigma^{2}+\Lambda_{1}+B_{i,j+1}}X_{i,j}, & i\neq N||j\neq n-1,\\ X_{N,n-1}, & i=N\&\&j=n-1,\end{cases} \tag{13}$$

where $i=1,2\ldots N,\ j=1,2,\ldots,n-1,$ *and Appendix 5.1 contains the iterated expression of* $B_{i,j+1}$.

The investor's optimal sizes of submitted orders in the ETC are

$$z_{i,n}^{*}=\begin{cases}\left(1-\frac{(1-p)\Lambda_{2}}{2p(B_{i+1,1}+\alpha\tilde{\sigma}^{2})}\right)X_{i,n}, & N>i>\bar{i},\\ 0, & i\leq\bar{i}||i=N,\end{cases} \tag{14}$$

where $i=1,2\ldots N$; *Appendix 5.1 contains expressions of* $\bar{i}$ *and* $B_{i+1,1}$.

Proposition 3.1 reveals the optimal liquidation strategy for multiple trading days. In (13), $B_{i,j+1}$ denotes the pressure to liquidate at the next stage that is the coefficient of the value function (minimized execution cost) at $t_{i,j+1}$. Intuitively the investor needs to trade-off between liquidating more in the current stage or retaining the position until the next stage. The investor's optimal sizes of submitted orders in the RTS are a linear function of the inventory. Next, we analyze the investor's optimal liquidation strategy in the ETC, that is, (14). Whether the investor submits an order in the ETC depends on the current trading day i, which is because submitting an ETC order too early exposes the

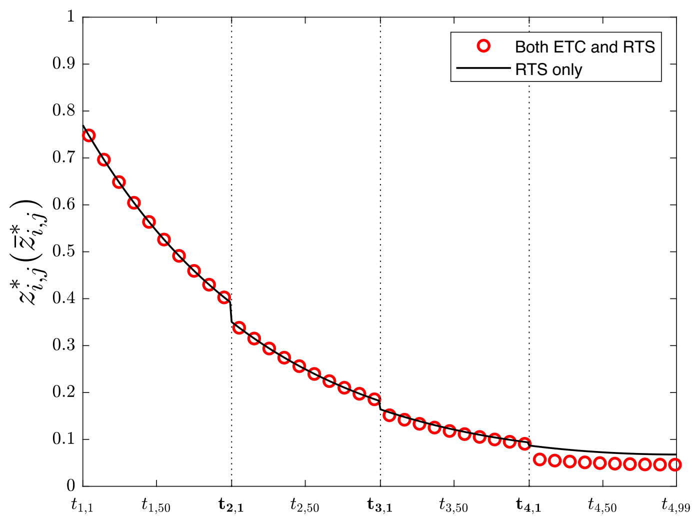

Fig. 2 The optimal sizes of submitted orders in the RTS

investor's the trading intention too early. If the current trading day is later than the critical value and not the last trading day ($N > i > \bar{i}$), then the investor trades in the ETC ($z^*_{i,n} > 0$). If the current trading day is earlier than or equal to the critical value ($i \leq \bar{i}$) or the last trading day ($i = N$), the investor does not trade in the ETC. Finally, we observe the optimal liquidation strategy via deterministic simulation.

Let the parameters in Fig. 2 be $X_{1,1} = 100$, $N = 4$, $n = 100$, $p = 20\%$, $\Lambda_1 = 225$, $\Lambda_2 = 1$, $\alpha = 5$, $\sigma = 0.05$ and $\tilde{\sigma} = 0.2$. The red circle line[11] and black line correspond to the optimal sizes of submitted orders with and without the ETC, respectively.[12] The optimal sizes of submitted orders in the ETC corresponding to Fig. 2 are $z^*_{1,n} = 0$, $z^*_{2,n} = 0.73$, $z^*_{3,n} = 2.16$, and $z^*_{4,n} = 0$, where all orders in the ETC are executed. We can determine the critical trading day $\bar{i} = 1$ in Fig. 2. Therefore, the investor submits orders in the ETC on the second and third trading days only. Thereafter, we investigate the optimal sizes of the submitted orders in the RTS. On the third and fourth trading days (from $t_{3,1}$ to $t_{4,99}$), the optimal sizes of submitted orders with the ETC (red circle line) are smaller than those without the ETC (black line) because if the orders in the ETC are executed, the investor needs to decrease the sizes of the orders in the RTS.

Execution costs with and without ETC

One of the goals of introducing the ETC in the Nasdaq, the SSE STAR and the SZSE ChiNext is to satisfy investors' trading demands. In this paper, the investor's trading demand is to reduce the execution cost. We define the execution costs saved by the ETC for the investor as

$$\Delta v := v_{1,1}\left(X_{1,1}, b_1 | z_{i,n} = 0\right) - v_{1,1}\left(X_{1,1}, b_1\right) = \inf_{\vec{z}_{1,1} \in \mathbb{A}_{1,1}} \{J_{1,1} | z_{i,n} = 0\} - \inf_{\vec{z}_{1,1} \in \mathbb{A}_{1,1}} J_{1,1}, \tag{15}$$

[11] To prevent overlapping, we draw a red circle for every ten points.

[12] When the ETC does not exist (RTS only), the sizes of submitted orders in the ETC are 0, that is, $\bar{z}_{i,n} = 0$. Appendix 5.4 contains the optimal liquidation strategy without the ETC, the proof of which is similar to Proposition 3.1.

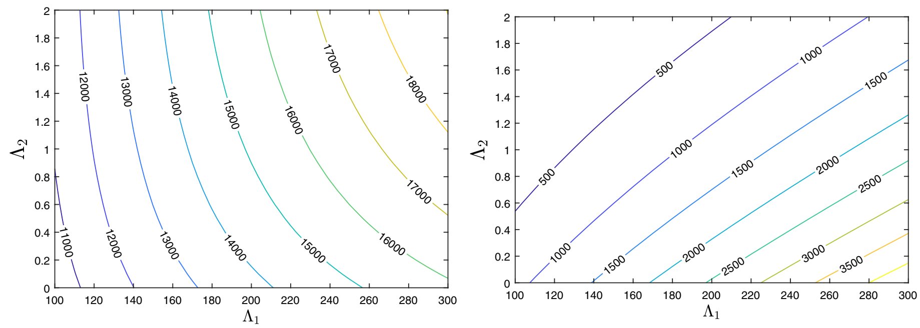

Fig. 3 Value function (minimized execution cost)

where $i = 1, 2 \ldots N$, and $v_{1,1}(X_{1,1}, b_1 | z_{i,n} = 0)$ is the value function without the ETC. To analyze whether the ETC satisfies the investor's trading demand, we illustrate $v_{1,1}(X_{1,1}, b_1)$ and Δv in Fig. 3 via deterministic simulation.

The parameters in Fig. 3, except for Λ_1, Λ_2 and $p = 50\%$, are the same as those in Fig. 2. Panel (a) illustrates the value function with the ETC, that is, $v_{1,1}(X_{1,1}, b_1)$. Panel (b) illustrates the difference in the value function without and with the ETC, that is, Δv. The vertical and horizontal axes represent the information price impact coefficient $\Lambda_2 \in [0, 2]$ and liquidity price impact coefficient $\Lambda_1 \in [100, 300]$, respectively. In Panel (a), the value function with the ETC increases with the liquidity price impact coefficient Λ_1 and information price impact coefficient Λ_2. In Panel (b), Δv is always greater than zero; thus, introducing the ETC saves execution costs for the investor. Moreover, the difference in the value function without and with the ETC increases as the liquidity price impact coefficient Λ_1 increases and the information price impact coefficient Λ_2 decreases. The reason for this phenomenon is that the liquidity price impact is the cost incurred in the RTS and the information price impact is the cost incurred in the ETC.

Finally, we examine the effect of the number of trading days on the value function (minimized execution cost) and draw a confusion in Corollary 3.1.

Corollary 3.1 *Provided that the other conditions are maintained, the value function* $v_{1,1}$ *decreases as the number of trading days N increases.*

Extreme price movements with and without the ETC

Fast liquidation consumes large amounts of liquidity, and this action is prone to extreme price movements in the market. To evaluate the mitigating effect that is produced by the ETC on extreme price movements, we construct an index *EPM* to investigate the extreme price movements that the investor's fast liquidation exerts on the market:

$$EPM = \sum_{q=1}^{N} \sum_{l=1}^{n-1} \left(\Lambda_1 z^*_{q,l} \right)^3. \tag{16}$$

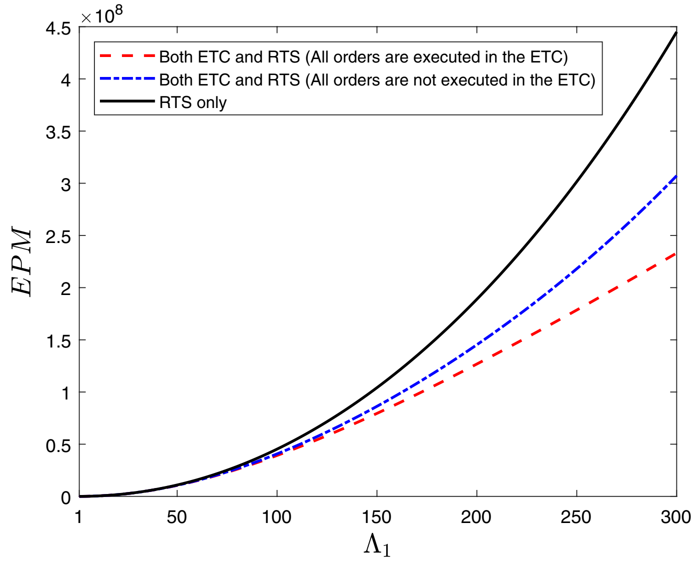

Fig. 4 Extreme price movements and liquidity price impact coefficient

Brogaard et al. (2018) empirically identifies the *EPM* by labeling all intervals that belong to the 99.9th percentile of the return distribution. In our model, if the *EPM* is determined using the price quantile, the *EPM* depends on random price fluctuations ε. To ensure the certainty and continuity of the *EPM*, we denote the *EPM* as the sum of the cubes of the liquidity price impact, where the cube captures the extremity of the impact.

We use a simple example to clarify the *EPM*: an investor needs to sell 1 million shares of a single asset in two trading days. We take two cases for the investor: in case 1, they sell 9 hundred thousand shares of the asset on the first day and 1 hundred thousand shares on the second day. In case 2, the sell half a million shares each day. Generally, case 1 is more likely to lead to extreme price movement in the market. Although the total number of shares of the asset in the two cases is the same, we should assign a high value to the *EPM* in case 1. To obtain a clear observation of the *EPM*, we compare the *EPM* with and without the ETC through deterministic simulation.

The parameters in Fig. 4, except for Λ_1 and $p = 50\%$, are the same as those in Fig. 2. Figure 4 presents three facts. First, if the ETC is introduced, no matter whether the orders in the ETC are executed, the *EPM* with the ETC is less than the *EPM* without the ETC. Second, with the liquidity in the RTS worsening, the mitigating effect of the ETC on liquidity impact is better (with an increasing Λ_1, the gap between the red or the blue dotted lines and the black line is greater). Third, the *EPM* with successful execution of orders in the ETC is far lower than the *EPM* with unfilled orders in the ETC. One of the goals of the SZSE ChiNext introducing the ETC is to enhance liquidity management methods. Therefore, when a quantity of unfilled orders exist in the ETC, the supervisor can submit orders earlier to match the surplus orders, which changes the *EPM*-Λ_1 line from the blue dotted line to the red dotted line. It is worth mentioning that the *EPM* with the ETC is lower than that without the ETC, even if the orders are not executed. This results from the fact that if the orders are not executed, the ETC encourages investors to reduce the sizes of the orders on the first few

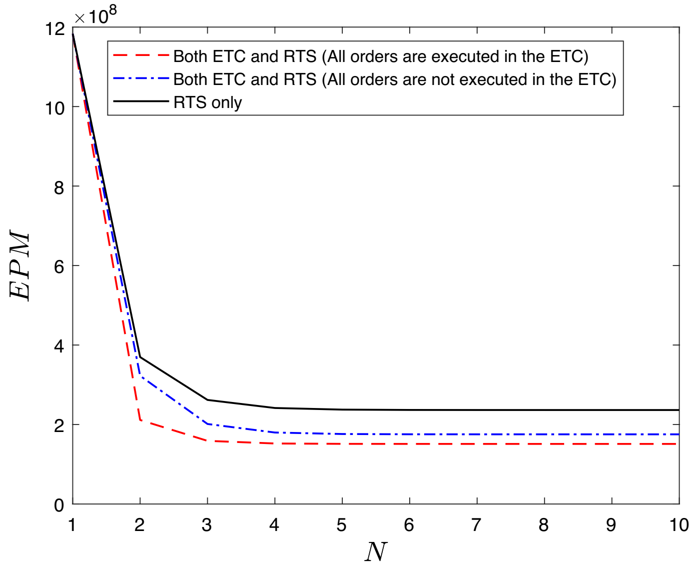

Fig. 5 Extreme price movements and number of trading days

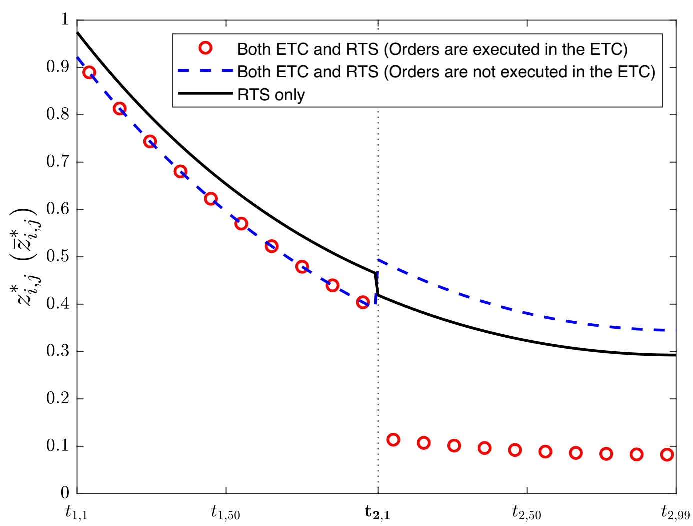

Fig. 6 The optimal sizes of submitted orders for two trading days in the RTS

trading days and to increase the size on the last few trading days in the RTS. This approach leads to the result that the investor reduces their extreme trading speed in the RTS. An example is displayed in Fig. 6.

Finally, we illustrate the relationship between the number of trading days and *EPM* in Fig. 5 via deterministic simulation.

Except for N and $p = 50\%$, the parameters in Fig. 5 are the same as those in Fig. 2. The vertical and horizontal axes represent extreme price movements *EPM* and the number of trading days N, respectively. According to Fig. 5, the introduction of the ETC reduces *EPM* regardless of whether the orders in the ETC are executed. Furthermore, *EPM* decreases as the number of trading days N increases, regardless of whether the ETC is introduced and whether orders are executed. This is becuase longer trading periods slow the investor's trading speed, which in turn reduces *EPM*.

Closed-form solutions for two trading days

Using the fixed-point iteration method, we can derive the closed-form solutions for two trading days. These solutions are summarized by the following proposition.

Proposition 4.1 *When $N = 2$, the optimal sizes of submitted orders in the RTS on the first day are*

$$z^*_{1,j} = \frac{\alpha\sigma^2 + B_{1,j+1}}{\alpha\sigma^2 + \Lambda_1 + B_{1,j+1}} X_{1,j}, \quad j = 1, 2, \ldots, n-1. \tag{17}$$

The optimal size of submitted order in the ETC on the first day is

$$z^*_{1,n} = \begin{cases} \left(1 - \frac{(1-p)\Lambda_2}{2p(c_2+\alpha\tilde{\sigma}^2)}\right) X_{1,n}, & \frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1, \\ 0, & \frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \le 1. \end{cases} \tag{18}$$

The optimal sizes of submitted orders in the RTS on the second day are

$$z^*_{2,j} = \left(1 - \frac{\sinh((n-j-1)c_1)}{\sinh((n-j)c_1)}\right) X_{2,j}, \quad j = 1, 2, \ldots, n-1. \tag{19}$$

The optimal size of submitted order in the ETC on the second day is $z^*_{2,n} = 0$. Appendix 5.2 contains the constants $B_{1,j+1}$, c_1 and c_2 in (17), (18) and (19).

Proposition 4.1 indicates the optimal liquidation strategy for two trading days. Then, we examine the influence of introducing the ETC on the liquidation strategy. In Corollary 4.1, we produce the difference between the optimal inventory $X^*_{1,j}$ with the ETC and the optimal inventory $\bar{X}^*_{1,j}$ without the ETC[13] on the first trading day. In Corollary 4.1, we also analyze the factors influencing the optimal proportion of the submitted order in the ETC to the inventory, that is, $\tilde{z}^*_{1,n} = \frac{z^*_{1,n}}{X_{1,n}}$. Intuitively, while the investor submitting an ETC order reduces the cost of liquidity price impacts and reduces overnight risk, on the other hand, there is a probability that the ETC order will not be executed and will cause an information price impact.

Corollary 4.1 *When $N = 2$, the optimal liquidation strategy has the following properties*:

(i) *the ETC motivates the investor to hold more inventory in the RTS on the first day, that is, $X^*_{1,j} \ge \bar{X}^*_{1,j}$.*
(ii) *For assets with a low information price impact coefficient, the investor should increase the proportion of orders submitted in the ETC, that is, $\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_2} \le 0$. In par-*

---
[13] $\bar{X}^*_{1,j}$ can be obtained from Appendix 5.4.

ticular, when the information price impact coefficient is zero ($\Lambda_2 = 0$), *they should submit the entire inventory in the ETC, that is,* $\tilde{z}^*_{1,n} = 1$.

(iii) *For assets with a high probability of execution in the ETC, the investor should increase the proportion of the orders submitted in the ETC, that is,* $\frac{\partial \tilde{z}^*_{1,n}}{\partial p} \geq 0$. *In particular, when all orders are executed in the ETC* ($p = 100\%$), *the investor should submit the entire inventory in the ETC, that is,* $\tilde{z}^*_{1,n} = 1$.

(iv) *For low-liquidity assets, the investor should increase the proportion of the submitted order in the ETC relative to the inventory, that is,* $\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_1} \geq 0$.

(v) *For high-volatility assets in the RTS* (*or overnight*), *the investor should increase the proportion of the submitted order in the ETC relative to the inventory, that is,* $\frac{\partial \tilde{z}^*_{1,n}}{\partial \sigma} \geq 0$ *and* $\frac{\partial \tilde{z}^*_{1,n}}{\partial \tilde{\sigma}} \geq 0$.

(vi) *An investor with a high risk-aversion coefficient should increase the proportion of the submitted order in the ETC relative to the inventory, that is,* $\frac{\partial \tilde{z}^*_{1,n}}{\partial \alpha} \geq 0$.

From Corollary 4.1 (i), if the ETC exists, the investor should reduce the trading speed in the RTS on the first trading day to wait for the trading opportunity in the ETC. From (ii), (iii), and (iv) of Corollary 4.1, for assets with a low information price impact coefficient, a high probability of execution of the orders in the ETC, less liquidity and high volatility, they should increase the proportion of the submitted order in the ETC relative to the inventory, and a high risk-aversion investor should also increase the proportion of the submitted order in the ETC relative to inventory.

For a clear view of the influence of the liquidation strategy exerted by the introduction of the ETC, we demonstrate the difference between the optimal sizes of submitted orders $z^*_{i,j}$ with the ETC and those $\bar{z}^*_{i,j}$ without the ETC through deterministic simulation, where $i = 1, 2$, $j = 1, 2 \ldots n - 1$ (Fig. 6).

The parameters in Fig. 6 are $X_{1,1} = 100$, $N = 2$, $n = 100$, $p = 50\%$, $\Lambda_1 = 150$, $\Lambda_2 = 1$, $\alpha = 5$, $\sigma = 0.05$ and $\tilde{\sigma} = 0.2$. The optimal sizes of submitted orders in the ETC corresponding to Fig. 6 are $z^*_{1,n} = 29.71$ and $z^*_{2,n} = 0$. Figure 6 shows two facts. First, the optimal sizes of submitted orders with the ETC on the first trading day are less than those without the ETC (the black line is higher than the blue dotted and the red circle lines[14] from $t_{1,1}$ to $t_{2,1}$), illustrating Corollary 4.1 (i). Second, from the beginning of the second trading day, the optimal sizes of submitted orders have jumps. We analyze jumps in the three lines. First, the red circle line undergoes a substantial jump down because successful execution of orders in the ETC alleviates the investor's liquidation stress in the RTS on the second trading day. Second, the blue dotted line undergoes a jump up because unsuccessful execution of orders in the ETC increases their liquidation stress in the RTS on the second trading day. Finally, the black line undergoes a slight jump down because the investor is inclined to quickly liquidate to lower the overnight risk until the end of the first trading day.

The model with permanent price impact

In "The main model" section, we assumed that only unexecuted orders in the ETC had permanent price impacts (information price impacts), that is, $\Lambda_2 b_i$. However, numerous studies have also considered permanent price impacts for executed orders, such as

[14] To prevent overlapping, we draw a red circle for every ten points.

Almgren and Chriss (2001), Obizhaeva and Wang (2013), and Tsoukalas et al. (2019). To generalize the model, we introduce permanent price impacts for executed orders in the ETC and RTS based on the model in "The main model" section. Specifically, we change the asset prices in the RTS from (4) to (20).

$$\hat{P}_{i,j} = \tilde{P}_{i,j} - \Lambda_1 z_{i,j} - \Lambda_2 \sum_{q=1}^{i-1} \left(z_{q,n} - a_{q,n} \right) - \gamma_1 \left(\sum_{q=1}^{i-1} \sum_{l=1}^{n-1} z_{q,l} + \sum_{l=1}^{j} z_{i,l} \right) - \gamma_2 \sum_{q=1}^{i-1} a_{q,n}, \tag{20}$$

where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n-1$. The first three terms of (20) agree with (4). The fourth term $\gamma_1 \big(\sum_{q=1}^{i-1} \sum_{l=1}^{n-1} z_{q,l} + \sum_{l=1}^{j} z_{i,l} \big)$ is the permanent price impact incurred by the executed orders in the RTS. γ_1 is a constant greater than zero and denotes the coefficient of permanent price impact in the RTS. $\sum_{q=1}^{i-1} \sum_{l=1}^{n-1} z_{q,l} + \sum_{l=1}^{j} z_{i,l}$ are the sizes of the executed (submitted) orders in the RTS before $t_{i,j}$ (including $t_{i,j}$). The fifth term $\gamma_2 \sum_{q=1}^{i-1} a_{q,n}$ is the permanent price impact owing to executed orders in the ETC. γ_2 is a constant greater than zero indicating the coefficient of the permanent price impact caused by executed orders in the ETC, and $\sum_{q=1}^{i-1} a_{q,n}$ are the sizes of executed orders in the ETC before $t_{i,j}$.

Furthermore, we change the asset price in the ETC from (5) to (21).

$$\hat{P}_{i,n} = \tilde{P}_{i,n} - \Lambda_2 \sum_{q=1}^{i-1} \left(z_{q,n} - a_{q,n} \right) - \gamma_1 \sum_{q=1}^{i} \sum_{l=1}^{n-1} z_{q,l} - \gamma_2 \sum_{q=1}^{i-1} a_{q,n}, \qquad i = 1, 2, \ldots, N. \tag{21}$$

The first two terms in (21) agree with (5). The third term $\gamma_1 \sum_{q=1}^{i} \sum_{l=1}^{n-1} z_{q,l}$ is the permanent price impact caused by the executed orders in the RTS before $t_{i,n}$. The fourth term $\gamma_2 \sum_{q=1}^{i-1} a_{q,n}$ is the permanent price impact caused by executed orders in the ETC before $t_{i,n}$. Note that owing to the fixed price, orders submitted by the investor in the ETC will not have any price impact on the current price.

Here, the value function is related not only to the current inventory and the cumulative sizes of unfilled orders in the ETC but also to the sizes of executed orders in the ETC and RTS. For convenience, we denote the size of executed orders in the RTS before $t_{i,j}$ by $g_{i,j} := \sum_{q=1}^{i-1} \sum_{l=1}^{n-1} z_{q,l} + \sum_{l=1}^{j-1} z_{i,l}$ and the size of the executed orders in the ETC before $t_{i,n}$ by $h_i := \sum_{q=1}^{i-1} a_{q,n}$. The value function $\hat{v}_{i,j}$ at $t_{i,j}$ is

$$\hat{v}_{i,j} \left(X_{i,j}, b_i, g_{i,j}, h_i \right) := \inf_{\bar{z}_{i,j} \in \mathbb{A}_{i,j}} J_{i,j}, \tag{22}$$

where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n$. The rest of the settings in this section are consistent with "The main model" section. On the assumption that executed orders would cause permanent price impacts, we obtain recursive solutions to the optimal liquidation strategy when the permanent price impact coefficient in the RTS is small ($\gamma_1 < 2(1-p)\alpha\tilde{\sigma}^2/p$), and we present this in Proposition 5.1 (for the proof, see Appendix 4).

Proposition 5.1 *On the assumption that executed orders would cause permanent price impacts and the permanent price impact coefficient of the RTS is small ($\gamma_1 < 2(1-p)\alpha\tilde{\sigma}^2/p$), the investor's optimal sizes of submitted orders in the RTS are*

$$\hat{z}^*_{i,j} = \begin{cases} \frac{2\hat{B}_{i,j+1}+2\alpha\sigma^2-\gamma_1}{2\left(\hat{B}_{i,j+1}+\alpha\sigma^2+\Lambda_1\right)}X_{i,j}, & i \neq N||j \neq n-1, \\ X_{N,n-1}, & i = N\&\&j = n-1, \end{cases} \tag{23}$$

where $i = 1, 2 \ldots N$, $j = 1, 2, \ldots, n-1$; Appendix 5.3 contains the iterated expression of $\hat{B}_{i,j+1}$.

The investor's optimal sizes of submitted orders in the ETC are

$$\hat{z}^*_{i,n} = \begin{cases} X_{i,n}, & \bar{\Lambda}_s(i) \geq \Lambda_2, \\ \frac{p(2\hat{B}_{i+1,1}+2\alpha\tilde{\sigma}^2-\gamma_2)-(1-p)\Lambda_2}{2p\left(\hat{B}_{i+1,1}+\alpha\tilde{\sigma}^2-\gamma_2\right)}X_{i,n}, & \bar{\Lambda}_b(i) \geq \Lambda_2 > \bar{\Lambda}_s(i), \\ 0, & \bar{\Lambda}_b(i) < \Lambda_2, \end{cases} \tag{24}$$

where $i = 1, 2 \ldots N-1$; Appendix 5.3 contains the expressions of $\hat{B}_{i+1,1}$, $\bar{\Lambda}_s(i)$ and $\bar{\Lambda}_b(i)$. Furthermore, the investor will not submit ETC orders on the last day to ensure that the liquidation is able to be completed, that is, $\hat{z}^*_{N,n} = 0$.

Proposition 5.1 indicates the optimal liquidation strategy for multiple trading days with permanent price impacts of executed orders. The technical reason that the permanent price impact coefficient in the RTS is small is that $\hat{B}_{i,j+1} > 0$ should be valid, and further, we have downward convexity, that is, (84). Moreover, the economic implication is to prevent arbitrage, that is, no-dynamic-arbitrage principle (Gatheral 2010). For example, an investor buys an asset in the RTS, exploits the permanent price impact to drive up the closing price, and then sells it in the ETC to make a profit. Consistent with Proposition 3.1, the investor's optimal size of submitted orders in the RTS is also a linear function of inventory size $X_{i,j}$, except that it only varies in coefficients. The investor's optimal size of submitted orders in the ETC is a piecewise linear function of inventory but with one additional case of submitting the entire inventory into the ETC in comparison to Proposition 3.1, that is, $\hat{z}^*_{i,n} = X_{i,n}$. This is because orders executed in the ETC do not have impacts on current prices, but they do on future prices. If the liquidation is completed in the current ETC, the costs associated with permanent price impact for the order executed in the current ETC can be avoided.

Conclusion and future research

We construct a model for an institutional investor who liquidates using the ETC over multiple trading days, from which we obtain the closed-form solutions for two trading days and recursive solutions for multiple trading days.

We obtain conclusions for an institutional investor that needs to liquidate a large block in a short time. First, the investor can reduce the execution costs by trading in the ETC, and the costs saved by the ETC increase as the liquidity price and the information price impact coefficients increases and decreases, respectively. Second, under the same conditions, for assets with a low information price impact coefficient, high probability of executed order in the ETC, low liquidity, and high volatility, investors should increase the ratio of the orders in the ETC relative to the inventory. Third, if the investor is high risk averse, they should increase the ratio of the orders in the ETC relative to the inventory. Fourth, in the model for two trading days, the investor should slow the trading speed appropriately in the RTS and wait for the trading opportunity in the ETC on the first trading day. Fifth, a critical trading day exists in the multiple trading days model, and the investor should not trade in the ETC until the critical trading day and on the last trading day; the investor should submit orders in the ETC after the critical trading day (except the last trading day). Sixth, as the number of trading days increases, the investor has more trading opportunities, and therefore the value function (minimized execution cost) decreases.

We also obtain conclusions for supervisors. First, the ETC satisfies the trade demands of an investor who needs to liquidate a large block. Second, regardless of whether the orders in the ETC are executed, extreme price movements caused by fast liquidation in the market can be reduced. Third, when the supervisor finds many unfilled orders in the ETC, they can match the orders earlier to prevent extreme price movements in the market. Fourth, as the number of trading days increases, extreme price movements caused by fast liquidation decrease.

This paper has three practical implications. First, to avoid prematurely revealing the investor's trading intention by unfilled orders in the ETC, they should not trade in the ETC on the front few trading days. Second, the investor should adjust the sizes of their orders submitted in the ETC according to the features of the asset, such as liquidity and volatility. Third, introducing the ETC can reduce liquidation costs and extreme price movements.

This paper offers four suggestions for future research. First, we assumed only two scenarios exist for order execution in ETC: full execution and nonexecution. Future research could extend this assumption to the case of partial order execution. Second, we assumed that the market environment was exogenously given, for example, the liquidity price impact coefficient. Future research could further consider the equilibrium model with multiple investors to endogenize these parameters. Third, this paper assumed that liquidity price impacts (temporary price impacts) will disappear completely in the next period. Future research can attempt to obtain the optimal liquidation strategy with the ETC when temporary price impacts gradually fade out. Fourth, we assumed that price impacts are linear functions of order volumes. Future research could extend the price impact to be nonlinear.

Appendix 1: Intuition for liquidity price impact

We follow Obizhaeva and Wang (2013) in linking linear price impacts to the limit order book, and display the limit order book in Fig. 7.

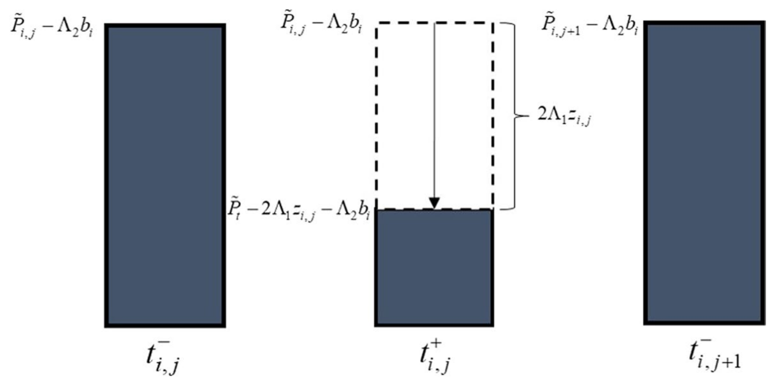

Fig. 7 Block limit order book

In Fig. 7, the height of each block order book indicates the price and the width ($\frac{1}{2\Lambda_1}$) indicates the liquidity available at the price level. $t^-_{i,j}$ and $t^+_{i,j}$ denote the start and end moments of stage $t_{i,j}$, respectively. The best selling price at the stage $t^-_{i,j}$ is $\widetilde{P}_{i,j}-\Lambda_2 b_i$. At the stage $t^+_{i,j}$, the investor sells the asset along the order book and brings the price down to $\widetilde{P}_{i,j}-2\Lambda_1 z_{i,j}-\Lambda_2 b_i$. At the stage $t^-_{i,j+1}$, the best price of the order book rebounds to $\widetilde{P}_{i,j+1}-\Lambda_2 b_i$. The order book at the stage $t^+_{i,j}$ shows that the investor's average trading price $P_{i,j}=\frac{(\widetilde{P}_{i,j}-\Lambda_2 b_i)+(\widetilde{P}_{i,j}-2\Lambda_1 z_{i,j}-\Lambda_2 b_i)}{2}=\widetilde{P}_{i,j}-\Lambda_1 z_{i,j}-\Lambda_2 b_i$.

Appendix 2: Proof of Proposition 3.1 and Corollary 3.1

Appendix 2.1: Proof of Proposition 3.1

We divide the proof of Proposition 3.1 into three steps. In step 1, we derive the value function $v_{N,n-1}(X_{N,n-1},b_N)$ at the endpoint $t_{N,n-1}$; in step 2, we derive the optimal liquidation strategy $z^*_{i,j}$ and the value function $v_{i,j}(X_{i,j},b_i)$ in the RTS; and in step 3, we derive the optimal liquidation strategy $z^*_{i,n}$ and the value function $v_{i,n}(X_{i,n},b_i)$ in the ETC.

Step 1. We derive the value function $v_{N,n-1}(X_{N,n-1},b_N)$ at the endpoint $t_{N,n-1}$. Because of the execution uncertainty in the ETC, the investor does not trade at $t_{N,n}$ in the ETC to guarantee the completion of liquidation, that is, $z^*_{N,n}=0$. Then, the investor must submit their entire inventory at the last stage of the RTS, that is, $z^*_{N,n-1}=X_{N,n-1}$. Substituting (4), (10), (11) and $z^*_{N,n-1}=X_{N,n-1}$ into (12), the value function at the endpoint is:

$$v_{N,n-1}(X_{N,n-1},b_N)=\left(\tilde{P}_{N,n-1}-P_{N,n-1}\right)X_{N,n-1}=\Lambda_1 X^2_{N,n-1}+\Lambda_2 b_N X_{N,n-1}. \quad (25)$$

Then, we denote the coefficient of the quadratic term in the value function at $t_{N,n-1}$ by $B_{N,n-1}$:

$$v_{N,n-1}(X_{N,n-1},b_N)=B_{N,n-1}X^2_{N,n-1}+\Lambda_2 b_N X_{N,n-1}, \quad (26)$$

where $B_{N,n-1}=\Lambda_1$.

Step 2. We derive the optimal liquidation strategy $z^*_{i,j}$ and the value function $v_{i,j}(X_{i,j},b_i)$ in the RTS. We suppose that $v_{i,j+1}(X_{i,j+1},b_i)=B_{i,j+1}X^2_{i,j+1}+\Lambda_2 b_i X_{i,j+1}$; then, we have the value function in the RTS:

$$
\begin{aligned}
v_{i,j}(X_{i,j},b_i) &= \inf_{z_{i,j}\le X_{i,j}} \Lambda_1 z_{i,j}^2 + \alpha\sigma^2\left(X_{i,j}-z_{i,j}\right)^2 + \Lambda_2 b_i z_{i,j} + v_{i,j+1}(X_{i,j+1},b_i) \\
&= \inf_{z_{i,j}\le X_{i,j}} \Lambda_1 z_{i,j}^2 + \left(\alpha\sigma^2 + B_{i,j+1}\right)\left(X_{i,j}-z_{i,j}\right)^2 + \Lambda_2 b_i X_{i,j} \\
&:= \inf_{z_{i,j}\le X_{i,j}} \tilde{v}_{i,j}(z_{i,j}),
\end{aligned} \tag{27}
$$

where $i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, n-1$. Taking partial derivatives $\tilde{v}_{i,j}$ with respect to $z_{i,j}$, we have

$$
\frac{\partial \tilde{v}_{i,j}}{\partial z_{i,j}} = 2\left(\alpha\sigma^2 + \Lambda_1 + B_{i,j+1}\right)z_{i,j} - 2\left(\alpha\sigma^2 + B_{i,j+1}\right)X_{i,j}, \tag{28}
$$

$$
\frac{\partial^2 \tilde{v}_{i,j}}{\partial z_{i,j}^2} = 2\left(\alpha\sigma^2 + \Lambda_1 + B_{i,j+1}\right) > 0, \tag{29}
$$

where $B_{i,j+1} \ge 0$ (see below for the proof). $\frac{\partial^2 \tilde{v}_{i,j}}{\partial z_{i,j}^2} > 0$ means convex downward. We calculate $\frac{\partial \tilde{v}_{i,j}}{\partial z_{i,j}} = 0$. Then, we derive the optimal sizes of the submitted orders at $t_{i,j}$:

$$
z_{i,j}^* = \frac{\alpha\sigma^2 + B_{i,j+1}}{\alpha\sigma^2 + \Lambda_1 + B_{i,j+1}} X_{i,j}, \qquad j = 1, 2, \ldots, n-1. \tag{30}
$$

Substituting (30) into (27), we have

$$
v_{i,j}(X_{i,j},b_i) = \frac{\Lambda_1\left(\alpha\sigma^2 + B_{i,j+1}\right)}{\alpha\sigma^2 + \Lambda_1 + B_{i,j+1}} X_{i,j}^2 + \Lambda_2 b_i X_{i,j} = B_{i,j} X_{i,j}^2 + \Lambda_2 b_i X_{i,j}, \tag{31}
$$

where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n-1$ and

$$
B_{i,j} = \frac{\Lambda_1\left(\alpha\sigma^2 + B_{i,j+1}\right)}{\alpha\sigma^2 + \Lambda_1 + B_{i,j+1}}. \tag{32}
$$

We know from (32) that if $B_{i,j+1} > 0$, then $B_{i,j} > 0$.

Step 3. We derive the optimal liquidation strategy $z_{i,n}^*$ and the value function $v_{i,n}(X_{i,n}, b_i)$ in the ETC. We suppose that $v_{i+1,1}(X_{i+1,1}, b_{i+1}) = B_{i+1,1}X_{i+1,1}^2 + \Lambda_2 b_{i+1} X_{i+1,1}$; then, we derive the value function:

$$
\begin{aligned}
v_{i,n}(X_{i,n},b_i) &= \inf_{0\le z_{i,n}\le X_{i,n}} \mathbb{E}_{i,n}[\alpha\tilde{\sigma}^2(X_{i,n}-a_{i,n})^2 + \Lambda_2 b_i a_{i,n} + v_{i+1,1}(X_{i+1,1},b_{i+1})] \\
&= \inf_{0\le z_{i,n}\le X_{i,n}} p\left(\left(B_{i+1,1}+\alpha\tilde{\sigma}^2\right)(X_{i,n}-z_{i,n})^2 + \Lambda_2 b_i X_{i,n}\right) + (1-p)\left(\left(B_{i+1,1}+\alpha\tilde{\sigma}^2\right)X_{i,n}^2 + \Lambda_2\left(b_i + z_{i,n}\right)X_{i,n}\right).
\end{aligned} \tag{33}
$$

Given $0 \le z_{i,n} \le X_{i.n}$, we derive the optimal solution to (33):

$$
z_{i,n}^* = \begin{cases} \left(1 - \frac{(1-p)\Lambda_2}{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}\right)X_{i,n}, & \frac{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1, \\ 0, & \frac{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \le 1. \end{cases} \tag{34}
$$

Substituting (34) into (33), we derive the value function at $t_{i,n}$:

$$
v_{i,n}(X_{i,n},b_i) = B_{i,n}X_{i,n}^2 + \Lambda_2 b_i X_{i,n}, \tag{35}
$$

where $i = 1, 2, \ldots, N-1$ and

$$B_{i,n} = \begin{cases} \frac{(1-p)\left(4p(B_{i+1,1}+\alpha\tilde{\sigma}^2)^2+4p\Lambda_2(B_{i+1,1}+\alpha\tilde{\sigma}^2)-(1-p)\Lambda_2^2\right)}{4p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}, & \frac{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1, \\ B_{i+1,1} + \alpha\tilde{\sigma}^2, & \frac{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1. \end{cases} \tag{36}$$

To simplify the description, we prove the existence of a critical trading day $\bar{i}$ by Lemma B.1. If the current trading day is less than $\bar{i}$ or is the last trading day ($i = N$), then the investor does not trade in the ETC ($z^*_{i,n} = 0$). If the current trading day is greater than $\bar{i}$ and is not the last trading day ($N > i > \bar{i}$), then the investor trades in the ETC ($z^*_{i,n} > 0$).

Lemma B.1 *If $z^*_{i,n} = 0$, then $z^*_{i-1,n} = 0$, where $i = 2, \ldots, N-1$.*

Proof
Let $\tilde{X} > 0$ and $\tilde{b} > 0$ be constant. From (31), we have

$$v_{i+1,1}(\tilde{X}, \tilde{b}) = B_{i+1,1}\tilde{X}^2 + \Lambda_2\tilde{b}\tilde{X}, \tag{37}$$

$$v_{i,1}(\tilde{X}, \tilde{b}) = B_{i,1}\tilde{X}^2 + \Lambda_2\tilde{b}\tilde{X}. \tag{38}$$

Because (12), we have $v_{i,1}(\tilde{X}, \tilde{b}) \leq v_{i+1,1}(\tilde{X}, \tilde{b})$. Combining (37) with (38), we have

$$v_{i,1}(\tilde{X}, \tilde{b}) - v_{i+1,1}(\tilde{X}, \tilde{b}) = (B_{i,1} - B_{i+1,1})\tilde{X}^2 \leq 0. \tag{39}$$

By (39), we have $B_{i,1} \leq B_{i+1,1}$. Thus, if $\frac{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1$, then $\frac{2p(B_{i,1}+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1$. In combination with (34), we have Lemma B.1. □

By Lemma B.1, we define the maximal trading day $\bar{i}$ when the investor does not trade in the ETC as

$$\bar{i} = \max\left(\left\{i = 1, 2, \ldots, N-1 \middle| \frac{2p(B_{i+1,1} + \alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1\right\} \cup 0\right). \tag{40}$$

Given (40) and $X_{N+1,1} = 0$, we can reorganize (34) as

$$z^*_{i,n} = \begin{cases} \left(1 - \frac{(1-p)\Lambda_2}{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}\right)X_{i,n}, & N > i > \bar{i}, \\ 0, & i \leq \bar{i} || i = N, \end{cases} \tag{41}$$

where $i = 1, 2, \ldots, N$.

Appendix 2.2: Proof of Corollary 3.1

Recall $b_i := \sum_{q=1}^{i-1}\left(z_{q,n} - a_{q,n}\right)$, so we derive $b_1 = 0$. Substitute $b_1 = 0$ into (31), and we have $v_{1,1} = B_{1,1}X_{1,1}^2$. We use $B_{i,j}(N)$ to denote that $B_{i,j}$ varies with the number of trading days N. From the recursive expression for $B_{i,j}$ in Appendix 5.1, we derive $B_{1,1}(N) = B_{i+1,1}(N+i)$, where i is an integer. Combining this with (39), we obtain

$B_{1,1}(N) = B_{i+1,1}(N+i) > B_{1,1}(N+i)$. Hence, both $v_{1,1}$ and $B_{1,1}(N)$ decrease as N increases.

Appendix 3: Proofs of Proposition 4.1 and Corollary 4.1

Appendix 3.1: Proof of Proposition 4.1

We divide the proof of proposition 4.1 into three steps. In step 1, we derive the optimal liquidation strategy $z^*_{2,j}$ on the second day and value function $v_{2,1}\left(X_{2,1}, b_2\right)$. In step 2, we derive the optimal liquidation strategy $z^*_{1,n}$ on the first day in the ETC and value function $v_{1,n}(X_{1,n}, b_1)$. In step 3, we derive the optimal liquidation strategy $z^*_{1,j}$ in the RTS on the first day.

Step 1. We derive the optimal liquidation strategy $z^*_{2,j}$ on the second trading day and the value function $v_{2,1}\left(X_{2,1}, b_2\right)$. The proof of step 1 is similar to Almgren and Chriss (2001). According to $X_{3,1} = 0$ and the execution uncertainty of the ETC, we have $z^*_{2,n} = 0$ and $X^*_{2,n} = 0$. Substituting (3), (4) and (10) into (11), we have

$$\begin{aligned} J_{2,1} &= \mathbb{E}_{2,1}[R_{2,1}] + \alpha \mathbb{E}_{2,1}\left[\sum_{q=2}^{n} \sigma^2_{2,q} X^2_{2,q}\right] \\ &= \mathbb{E}_{2,1}\left[\tilde{P}_{2,1} X_{2,1} - \sum_{q=1}^{n-1} P_{2,q} a_{2,q}\right] + \alpha\sigma^2 \sum_{q=2}^{n} X^2_{2,q} \\ &= \Lambda_1 \sum_{q=1}^{n-1} a^2_{2,q} + \Lambda_2 \left(z_{1,n} - a_{1,n}\right) \sum_{q=1}^{n-1} a_{2,q} + \alpha\sigma^2 \sum_{q=2}^{n} X^2_{2,q} \\ &= \Lambda_1 \sum_{q=1}^{n-1} (X_{2,q} - X_{2,q+1})^2 + \Lambda_2 b_2 \sum_{q=1}^{n-1} (X_{2,q} - X_{2,q+1}) + \alpha\sigma^2 \sum_{q=2}^{n} X^2_{2,q}. \end{aligned} \tag{42}$$

Taking partial derivatives $J_{2,1}$ with respect to $X_{2,j}$, we have

$$\frac{\partial J_{2,1}}{\partial X_{2,j}} = 4\Lambda_1 X_{2,j} - 2\Lambda_1 X_{2,j+1} - 2\Lambda_1 X_{2,j-1} + 2\alpha\sigma^2 X_{2,j}, \tag{43}$$

$$\frac{\partial^2 J_{2,1}}{\partial X^2_{2,j}} = 4\Lambda_1 + 2\alpha\sigma^2 > 0, \tag{44}$$

where $j = 2, 3, \ldots, n-1$. We know from $\frac{\partial^2 J_{2,1}}{\partial X^2_{2,j}} > 0$ that the optimal solution is $\frac{\partial J_{2,1}}{\partial X_{2,j}} = 0$. Reorganizing (43), we build the difference equations:

$$X_{2,j+1} - \left(2 + \frac{\alpha\sigma^2}{\Lambda_1}\right) X_{2,j} + X_{2,j-1} = 0. \tag{45}$$

According to the general solution to quadratic homogeneous difference equations, we have

$$
\begin{aligned}
X_{2,j}^{*} &= c_3\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1+\sqrt{\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1\right)^2-1}\right)^j + c_4\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1-\sqrt{\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1\right)^2-1}\right)^j \\
&= c_3 e^{j\cdot\ln\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1+\sqrt{\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1\right)^2-1}\right)} + c_4 e^{j\cdot\ln\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1-\sqrt{\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1\right)^2-1}\right)} \\
&= c_3 e^{j\cdot\operatorname{arcosh}\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1\right)} + c_4 e^{-j\cdot\operatorname{arcosh}\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1\right)} \\
&= c_3 e^{j\cdot c_1} + c_4 e^{-j\cdot c_1},
\end{aligned}
\tag{46}
$$

where $c_1 = \operatorname{arcosh}\left(\frac{\alpha\sigma^2}{2\Lambda_1}+1\right)$ and $j = 1, 2, 3, \ldots, n$. By the conditions ($X_{2,1}$ and $X_{2,n} = 0$), we can calculate c_3 and c_4:

$$
\begin{cases} c_3 e^{c_1} + c_4 e^{-c_1} = X_{2,1}, \\ c_3 e^{n\cdot c_1} + c_4 e^{-n\cdot c_1} = 0. \end{cases}
\tag{47}
$$

Solving the simultaneous equations (47), we have

$$
\begin{cases} c_3 = \frac{-e^{-n\cdot c_1}}{e^{(n-1)c_1} - e^{-(n-1)c_1}} X_{2,1}, \\ c_4 = \frac{e^{n\cdot c_1}}{e^{(n-1)c_1} - e^{-(n-1)c_1}} X_{2,1}. \end{cases}
\tag{48}
$$

Substituting (48) into (46), we have

$$
\begin{aligned}
X_{2,j}^{*} &= c_3 e^{j\cdot c_1} + c_4 e^{-j\cdot c_1} = \frac{-e^{-n\cdot c_1}}{e^{(n-1)c_1} - e^{-(n-1)c_1}} e^{j\cdot c_1} X_{2,1} + \frac{e^{n\cdot c_1}}{e^{(n-1)c_1} - e^{-(n-1)c_1}} e^{-j\cdot c_1} X_{2,1} \\
&= \frac{-e^{-(n-j)\cdot c_1}}{e^{(n-1)c_1} - e^{-(n-1)c_1}} X_{2,1} + \frac{e^{(n-j)\cdot c_1}}{e^{(n-1)c_1} - e^{-(n-1)c_1}} X_{2,1} \\
&= \frac{\sinh((n-j)c_1)}{\sinh((n-1)c_1)} X_{2,1}.
\end{aligned}
\tag{49}
$$

Combining (3), (49) with $z_{i,j} = a_{i,j}$, we have

$$
\begin{aligned}
z_{2,j}^{*} = a_{2,j}^{*} = X_{2,j}^{*} - X_{2,j+1}^{*} &= \frac{\sinh((n-j)c_1) - \sinh((n-j-1)c_1)}{\sinh((n-1)c_1)} X_{2,1} \\
&= \left(1 - \frac{\sinh((n-j-1)c_1)}{\sinh((n-j)c_1)}\right) X_{2,j}.
\end{aligned}
\tag{50}
$$

Substituting (42) and (49) into (12), we have

$$
\begin{aligned}
v_{2,1}(X_{2,1}, b_2) &= \inf_{\bar{z}_{2,1}\in\mathbb{A}_{2,1}} J_{2,1} \\
&= \Lambda_1 \sum_{q=1}^{n-1} (X_{2,q}^{*} - X_{2,q+1}^{*})^2 + \Lambda_2 b_2 \sum_{q=1}^{n-1} (X_{2,q}^{*} - X_{2,q+1}^{*}) + \alpha\sigma^2 \sum_{q=2}^{n} X_{2,q}^{*2} \\
&= \operatorname{csch}^2((n-1)c_1) \sinh\left(\frac{c_1}{2}\right)\left(\sinh\left(\frac{c_1}{2}\right) - \sinh\left(\frac{1}{2}(5-4n)c_1\right)\right)\Lambda_1 X_{2,1}^2 + \Lambda_2 b_2 X_{2,1} \\
&= c_2 X_{2,1}^2 + \Lambda_2 b_2 X_{2,1},
\end{aligned}
\tag{51}
$$

where $c_2 := \operatorname{csch}^2((n-1)c_1)\sinh\left(\frac{c_1}{2}\right)\left(\sinh\left(\frac{c_1}{2}\right) - \sinh\left(\frac{1}{2}(5-4n)c_1\right)\right)\Lambda_1$.

Step 2. We derive the optimal liquidation strategy $z^*_{1,n}$ in the ETC on the first trading day and the value function $v_{1,n}(X_{1,n}, b_1)$. Combining (2), (5), and (10) with (11), we have the relationship between $J_{1,n}$ and $J_{2,1}$:

$$J_{1,n} = \mathbb{E}_{1,n}[J_{2,1} + \Lambda_2 b_1 a_{1,n} + \alpha\tilde{\sigma}^2 X_{2,1}^2]. \tag{52}$$

We know that the unfilled order in the ETC before the first trading day is 0, that is, $b_1 = 0$. Then, substituting (1), (51) and (52) into (12), we have the Bellman equation of $v_{1,n}(X_{1,n}, 0)$:

$$\begin{aligned} v_{1,n}(X_{1,n},0) &= \inf_{0\le z_{1,n}\le X_{1,n}} \mathbb{E}_{1,n}[v_{2,1}(X_{2,1},b_2) + \alpha\tilde{\sigma}^2 X_{2,1}^2] \\ &= \inf_{0\le z_{1,n}\le X_{1,n}} p\Big(\alpha\tilde{\sigma}^2\big(X_{1,n}-z_{1,n}\big)^2 + v_{2,1}(X_{1,n}-z_{1,n},0)\Big) \\ &\quad + (1-p)\Big(\alpha\tilde{\sigma}^2 X_{1,n}^2 + v_{2,1}(X_{1,n},z_{1,n})\Big) \\ &= \inf_{0\le z_{1,n}\le X_{1,n}} p\Big(c_2+\alpha\tilde{\sigma}^2\Big)\big(X_{1,n}-z_{1,n}\big)^2 \\ &\quad + (1-p)\Big(\Big(c_2+\alpha\tilde{\sigma}^2\Big)X_{1,n}^2 + \Lambda_2 z_{1,n}X_{1,n}\Big). \end{aligned} \tag{53}$$

In combination with $0 \le z_{1,n} \le X_{1,n}$, we derive the optimal solution of (53):

$$z^*_{1,n} = \begin{cases} \Big(1 - \frac{(1-p)\Lambda_2}{2p(c_2+\alpha\tilde{\sigma}^2)}\Big)X_{1,n}, & \frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1, \\ 0, & \frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \le 1. \end{cases} \tag{54}$$

Substituting (54) into (53), we derive the value function $v_{1,n}(X_{1,n}, b_1)$:

$$v_{1,n}(X_{1,n}, b_1) = B_{1,n}X_{1,n}^2, \tag{55}$$

where

$$B_{1,n} = \begin{cases} \frac{(1-p)}{4p(c_2+\alpha\tilde{\sigma}^2)}\big(4pc_2^2 - (1-p)\Lambda_2^2 + 4p\alpha\tilde{\sigma}^2\Lambda_2 + 4p\alpha^2\tilde{\sigma}^4 + 4pc_2(\Lambda_2+2\alpha\tilde{\sigma}^2)\big), & \frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1, \\ c_2+\alpha\tilde{\sigma}^2, & \frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \le 1. \end{cases} \tag{56}$$

Step 3. we derive the optimal liquidation strategy $z^*_{1,j}$ in the RTS on the first trading day. The ideas for the proof of step 3 follow. First, the value function $v_{1,n}(X_{1,n}, 0)$ at the end-point $t_{1,n}$ is the quadratic expression of the inventory. Second, if the value function at $t_{1,j+1}$ is the quadratic expression of the inventory $v_{1,j+1}(X_{1,j+1}, 0) = B_{1,j+1}X_{1,j+1}^2$, then the value function at $t_{1,j}$ is also $v_{1,j}(X_{1,j}, 0) = B_{1,j}X_{1,j}^2$. Third, we derive the closed form of $z^*_{1,j}$.

First, we know from (55) that $v_{1,n}(X_{1,n}, 0)$ is the quadratic expression of the inventory. We suppose that $v_{1,j+1}(X_{1,j+1}, 0) = B_{1,j+1}X_{1,j+1}^2$, and then, the Bellman equation of the value function $v_{1,j}(X_{1,j}, 0)$ is

$$\begin{aligned} v_{1,j}(X_{1,j},0) &= \inf_{z_{1,j}\le X_{1,j}} v_{1,j+1}(X_{1,j+1},0) + \Lambda_1 z_{1,j}^2 + \alpha\sigma^2\big(X_{1,j}-z_{1,j}\big)^2 \\ &= \inf_{z_{1,j}\le X_{1,j}} B_{1,j+1}\big(X_{1,j}-z_{1,j}\big)^2 + \Lambda_1 z_{1,j}^2 + \alpha\sigma^2\big(X_{1,j}-z_{1,j}\big)^2 \\ &:= \inf_{z_{1,j}\le X_{1,j}} \tilde{v}_{1,j}(z_{1,j}), \end{aligned} \tag{57}$$

where $j = 1, 2, \ldots, n-1$. Taking partial derivatives $\tilde{v}_{1,j}$ with respect to $z_{1,j}$, we have

$$\frac{\partial \tilde{v}_{1,j}}{\partial z_{1,j}} = -2\left(\alpha\sigma^2 + B_{1,j+1}\right)X_{1,j} + 2\left(\alpha\sigma^2 + \Lambda_1 + B_{1,j+1}\right)z_{1,j}, \tag{58}$$

$$\frac{\partial^2 \tilde{v}_{1,j}}{\partial z_{1,j}^2} = 2\left(\alpha\sigma^2 + \Lambda_1 + B_{1,j+1}\right) > 0, \tag{59}$$

where $B_{1,j+1} \geq 0$ (see below for the proof). $\frac{\partial^2 \tilde{v}_{1,j}}{\partial z_{1,j}^2} > 0$ means convex downward. Reorganizing $\frac{\partial \tilde{v}_{1,j}}{\partial z_{1,j}} = 0$, we derive the optimal size of order at $t_{1,j}$:

$$z_{1,j}^* = \frac{\alpha\sigma^2 + B_{1,j+1}}{\alpha\sigma^2 + \Lambda_1 + B_{1,j+1}} X_{1,j}. \tag{60}$$

Substituting (60) into (57), we have

$$v_{1,j}(X_{1,j}, 0) = \frac{\Lambda_1\left(\alpha\sigma^2 + B_{1,j+1}\right)}{\alpha\sigma^2 + \Lambda_1 + B_{1,j+1}} X_{1,j}^2 = B_{1,j} X_{1,j}^2. \tag{61}$$

where

$$B_{1,j} = \frac{\Lambda_1\left(\alpha\sigma^2 + B_{1,j+1}\right)}{\alpha\sigma^2 + \Lambda_1 + B_{1,j+1}}. \tag{62}$$

We can find $B_{1,j} > 0$ if $B_{1,j+1} > 0$. Therefore, it holds that $B_{1,j} > 0$ for $j = 1, 2, \ldots, n-1$.

Finally, we derive the closed form of $z_{1,j}^*$. $f(x)$ is defined as

$$f(x) := \frac{\Lambda_1\left(\alpha\sigma^2 + x\right)}{\alpha\sigma^2 + \Lambda_1 + x}; \tag{63}$$

$f^{(j)}(x)$ denotes the j-th iteration of $f(x)$, where $f^{(0)}(x) = B_{1,n}$. We deduce $B_{1,j} = f^{(n-j)}(x)$ from the iteration. With the fixed-point iteration method ($f(x) = x$), we obtain its bridge function:

$$\varphi_1(x) = \frac{x + \frac{\alpha\sigma^2 + \sqrt{\alpha^2\sigma^4 + 4\Lambda_1\alpha\sigma^2}}{2}}{x + \frac{\alpha\sigma^2 - \sqrt{\alpha^2\sigma^4 + 4\Lambda_1\alpha\sigma^2}}{2}}. \tag{64}$$

We derive the inverse function of $\varphi_1(x)$:

$$\varphi_2(x) := \varphi_1^{-1}(x) = \frac{\alpha\sigma^2(1-x) + (x+1)\sqrt{\alpha\sigma^2(4\Lambda_1 + \alpha\sigma^2)}}{2(x-1)}. \tag{65}$$

We establish the function $g(x)$ as

$$g(x) := \varphi_1(f(\varphi_2(x))) = \left(1 + \frac{\alpha\sigma^2\left(\alpha\sigma^2 + \sqrt{\alpha\sigma^2(4\Lambda_1 + \alpha\sigma^2)}\right)}{2\Lambda_1^2} + \frac{2\alpha\sigma^2 + \sqrt{\alpha\sigma^2(4\Lambda_1 + \alpha\sigma^2)}}{\Lambda_1}\right)x. \tag{66}$$

By mathematical induction, we have $f^{(j)}(x) = \varphi_2(g^{(j)}(\varphi_1(x)))$, where integer $j \geq 1$. We have

$$\begin{aligned} B_{1,j} &= f^{(n-j)}(B_{1,n}) = \varphi_2(g^{(n-j)}(\varphi_1(B_{1,n}))) \\ &= \frac{c_5^n\left(2\alpha\sigma^2\Lambda_1 + \left(-\alpha\sigma^2 + \sqrt{\alpha\sigma^2(\alpha\sigma^2+4\Lambda_1)}\right)B_{1,n}\right) + c_5^j\left(-2\alpha\sigma^2\Lambda_1 + \left(\alpha\sigma^2 + \sqrt{\alpha\sigma^2(\alpha\sigma^2+4\Lambda_1)}\right)B_{1,n}\right)}{c_5^j\left(-\alpha\sigma^2 + \sqrt{\alpha\sigma^2(\alpha\sigma^2+4\Lambda_1)} - 2B_{1,n}\right) + c_5^n\left(\alpha\sigma^2 + \sqrt{\alpha\sigma^2(\alpha\sigma^2+4\Lambda_1)} + 2B_{1,n}\right)}, \end{aligned} \tag{67}$$

where $c_5 = 1 + \frac{\alpha\sigma^2\left(\alpha\sigma^2 + \sqrt{\alpha\sigma^2(4\Lambda_1+\alpha\sigma^2)}\right)}{2\Lambda_1^2} + \frac{2\alpha\sigma^2 + \sqrt{\alpha\sigma^2(4\Lambda_1+\alpha\sigma^2)}}{\Lambda_1}$, $j = 1, 2, \ldots, n$. Substituting (56) and (67) into (60), we have the closed form of $z_{1,j}^*$, that is, (17).

Appendix 3.2: Proof of Corollary 4.1

First, we prove $X_{1,j}^* \geq \bar{X}_{1,j}^*$. Substituting $\bar{z}_{1,n} = 0$ into (53), we derive the value function without the ETC

$$\bar{v}_{1,n}(X_{1,n}, 0) = (c_2 + \alpha\tilde{\sigma}^2)X_{1,n}^2 = \bar{B}_{1,n}X_{1,n}^2, \tag{68}$$

where $\bar{B}_{1,n} = c_2 + \alpha\tilde{\sigma}^2$. We have $B_{1,n} \leq \bar{B}_{1,n}$ by comparing (56) and (68). From (62), if $B_{1,j+1} \leq \bar{B}_{1,j+1}$, we have

$$B_{1,j} - \bar{B}_{1,j} = \frac{\Lambda_1\left(\alpha\sigma^2 + B_{1,j+1}\right)}{\alpha\sigma^2 + \Lambda_1 + B_{1,j+1}} - \frac{\Lambda_1\left(\alpha\sigma^2 + \bar{B}_{1,j+1}\right)}{\alpha\sigma^2 + \Lambda_1 + \bar{B}_{1,j+1}} \leq 0, \qquad j = 1, 2, \ldots n-1. \tag{69}$$

By mathematical induction, we have $B_{1,j} \leq \bar{B}_{1,j}$ for $j = 1, 2, \ldots, n$. Substituting (17) and $z_{1,j} = a_{1,j}$ into (3), we have the optimal inventory at $t_{1,j}$:

$$X_{1,j+1}^* = X_{1,j} - z_{1,j}^* = \frac{\Lambda_1}{\alpha\sigma^2 + \Lambda_1 + B_{1,j+1}}X_{1,j}, \qquad j = 1, 2, \ldots, n-1. \tag{70}$$

By iteration equations, we derive

$$X_{1,j}^* = \left(\prod_{q=2}^{j} \frac{\Lambda_1}{\alpha\sigma^2 + \Lambda_1 + B_{1,q}}\right)X_{1,1}, \qquad j = 1, 2, \ldots, n. \tag{71}$$

Analogously, we derive the optimal inventory at $t_{1,j}$ without the ETC:

$$\bar{X}_{1,j}^* = \left(\prod_{q=2}^{j} \frac{\Lambda_1}{\alpha\sigma^2 + \Lambda_1 + \bar{B}_{1,q}}\right)X_{1,1}, \qquad j = 1, 2, \ldots, n. \tag{72}$$

Combining (71), (72) with $B_{1,j} \leq \bar{B}_{1,j}$, we have

$$X_{1,j}^* - \bar{X}_{1,j}^* = \left(\prod_{q=2}^{j} \frac{\Lambda_1}{\alpha\sigma^2 + \Lambda_1 + B_{1,q}} - \prod_{q=2}^{j} \frac{\Lambda_1}{\alpha\sigma^2 + \Lambda_1 + \bar{B}_{1,q}}\right)X_{1,1} \geq 0, \tag{73}$$

where $j = 1, 2, \ldots, n$. Therefore, we derive $X_{1,j}^* \geq \bar{X}_{1,j}^*$.

Second, we prove that $\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_2} \leq 0$. From (18), we know that if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1$, then $\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_2} \leq 0$. Moreover, if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1$, then $\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_2} = 0$. We can conclude that $\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_2} \leq 0$. Substituting $\Lambda_2 = 0$ into (18), we have $\tilde{z}^*_{1,n} = 1$.

Third, we prove that $\frac{\partial \tilde{z}^*_{1,n}}{\partial p} \geq 0$. From (18), we know that if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1$, then $\frac{\partial \tilde{z}^*_{1,n}}{\partial p} \geq 0$. Additionally, if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1$, then $\frac{\partial \tilde{z}^*_{1,n}}{\partial p} = 0$. We can conclude that $\frac{\partial \tilde{z}^*_{1,n}}{\partial p} \geq 0$. Substituting $p = 100\%$ into (18), we have $\tilde{z}^*_{1,n} = 1$.

Fourth, we prove that $\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_1} \geq 0$. From (18), we know that if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1$, then

$$\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_1} = \frac{\partial \tilde{z}^*_{1,n}}{\partial c_2}\frac{\partial c_2}{\partial \Lambda_1} \geq 0, \tag{74}$$

and if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1$, then

$$\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_1} = 0. \tag{75}$$

We can conclude that $\frac{\partial \tilde{z}^*_{1,n}}{\partial \Lambda_1} \geq 0$.

Fifth, we prove that $\frac{\partial \tilde{z}^*_{1,n}}{\partial \sigma} \geq 0$ and $\frac{\partial \tilde{z}^*_{1,n}}{\partial \tilde{\sigma}} \geq 0$. From (18), we know that if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1$, then

$$\frac{\partial \tilde{z}^*_{1,n}}{\partial \sigma} = \frac{\partial \tilde{z}^*_{1,n}}{\partial c_2}\frac{\partial c_2}{\partial \sigma} \geq 0, \qquad \frac{\partial \tilde{z}^*_{1,n}}{\partial \tilde{\sigma}} \geq 0, \tag{76}$$

and if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1$, then

$$\frac{\partial \tilde{z}^*_{1,n}}{\partial \sigma} = \frac{\partial \tilde{z}^*_{1,n}}{\partial \tilde{\sigma}} = 0. \tag{77}$$

We can conclude that $\frac{\partial \tilde{z}^*_{1,n}}{\partial \sigma} \geq 0$, $\frac{\partial \tilde{z}^*_{1,n}}{\partial \tilde{\sigma}} \geq 0$.

Sixth, we prove that $\frac{\partial \tilde{z}^*_{1,n}}{\partial \alpha} \geq 0$. From (18), we know that if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1$, then

$$\frac{\partial \tilde{z}^*_{1,n}}{\partial \alpha} \geq 0, \tag{78}$$

and if $\frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1$, then

$$\frac{\partial \tilde{z}^*_{1,n}}{\partial \alpha} = 0. \tag{79}$$

We can conclude that $\frac{\partial \tilde{z}^*_{1,n}}{\partial \alpha} \geq 0$.

Appendix 4: Proof of Proposition 5.1

Similar to the proof of Proposition 3.1, we divide the proof of Proposition 5.1 into three steps. In step 1, we derive the value function $\hat{v}_{N,n-1}(X_{N,n-1}, b_N, g_{N,n-1}, h_N)$ at the endpoint $t_{N,n-1}$; in step 2, we derive the optimal liquidation strategy $\hat{z}^*_{i,j}$ and the value function $\hat{v}_{i,j}(X_{i,j}, b_i, g_{i,j}, h_i)$ in the RTS; and in step 3, we derive the optimal liquidation strategy $\hat{z}^*_{i,n}$ and the value function $\hat{v}_{i,n}(X_{i,n}, b_i, g_{i,n}, h_i)$ in the ETC.

Step 1. We derive the value function $\hat{v}_{N,n-1}(X_{N,n-1}, b_N, g_{N,n-1}, h_N)$ at the endpoint $t_{N,n-1}$. Because of the execution uncertainty in the ETC, the investor does not trade at $t_{N,n}$ in the ETC to guarantee the completion of liquidation, that is, $\hat{z}^*_{N,n} = 0$. Then, investor must submit their entire inventory at the last stage of the RTS, that is, $\hat{z}^*_{N,n-1} = \hat{X}_{N,n-1}$. Substituting (10), (11), (20) and $z^*_{N,n-1} = X_{N,n-1}$ into (22), the value function at the endpoint is

$$\begin{aligned}\hat{v}_{N,n-1}(X_{N,n-1}, b_N, g_{N,n-1}, h_N) &= \left(\tilde{P}_{N,n-1} - \hat{P}_{N,n-1}\right)X_{N,n-1} \\ &= \Lambda_1 X^2_{N,n-1} + \Lambda_2 b_N X_{N,n-1} + \gamma_1(g_{N,n-1} + X_{N,n-1})X_{N,n-1} \\ &\quad + \gamma_2 h_N X_{N,n-1}.\end{aligned} \tag{80}$$

Then, we denote the coefficient of the quadratic term in the value function at $t_{N,n-1}$ by $B_{N,n-1}$:

$$\hat{v}_{N,n-1}(X_{N,n-1}, b_N, g_{N,n-1}, h_N) = \hat{B}_{N,n-1}X^2_{N,n-1} + \Lambda_2 b_N X_{N,n-1} + \gamma_1 g_{N,n-1}X_{N,n-1} + \gamma_2 h_N X_{N,n-1}, \tag{81}$$

where $B_{N,n-1} = \Lambda_1 + \gamma_1 > \gamma_1/2$.

Step 2. We derive the optimal liquidation strategy $\hat{z}^*_{i,j}$ and the value function $\hat{v}_{i,j}(X_{i,j}, b_i, g_{i,j}, h_i)$ in the RTS. We suppose that $\hat{v}_{i,j+1}(X_{i,j+1}, b_i, g_{i,j+1}, h_i) = \hat{B}_{i,j+1}X^2_{i,j+1}$ $+(\Lambda_2 b_i + \gamma_1 g_{i,j+1} + \gamma_2 h_i)X_{i,j+1}$; then, we have the value function in the RTS:

$$\begin{aligned}\hat{v}_{i,j}(X_{i,j}, b_i, g_{i,j}, h_i) &= \inf_{z_{i,j}\le X_{i,j}} \Lambda_1 z^2_{i,j} + \alpha\sigma^2\left(X_{i,j} - z_{i,j}\right)^2 + \Lambda_2 b_i z_{i,j} + \gamma_1(g_{i,j} + z_{i,j})z_{i,j} + \gamma_2 h_i z_{i,j} \\ &\quad + \hat{v}_{i,j+1}(X_{i,j} - z_{i,j}, b_i, g_{i,j} + z_{i,j}, h_i) \\ &= \inf_{z_{i,j}\le X_{i,j}} (\Lambda_1 + \gamma_1)z^2_{i,j} + (\alpha\sigma^2 + \hat{B}_{i,j+1})(X_{i,j} - z_{i,j})^2 + \Lambda_2 b_i X_{i,j} + \gamma_1 g_{i,j} z_{i,j} \\ &\quad + \gamma_1(g_{i,j} + z_{i,j})(X_{i,j} - z_{i,j}) + \gamma_2 h_i X_{i,j} \\ &:= \inf_{z_{i,j}\le X_{i,j}} \bar{v}_{i,j}(z_{i,j}),\end{aligned} \tag{82}$$

where $i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, n-1$. Taking partial derivatives $\bar{v}_{i,j}$ with respect to $z_{i,j}$, we have

$$\frac{\partial \bar{v}_{i,j}}{\partial z_{i,j}} = X_{i,j}\left(\gamma_1 - 2\left(\hat{B}_{i,j+1} + \alpha\sigma^2\right)\right) + 2z_{i,j}\left(\hat{B}_{i,j+1} + \alpha\sigma^2 + \Lambda_1\right), \tag{83}$$

$$\frac{\partial^2 \bar{v}_{i,j}}{\partial z^2_{i,j}} = 2\left(\hat{B}_{i,j+1} + \alpha\sigma^2 + \Lambda_1\right) > 0, \tag{84}$$

where $\hat{B}_{i,j+1} \ge \gamma_1/2$ (see below for the proof). $\frac{\partial^2 \bar{v}_{i,j}}{\partial z^2_{i,j}} > 0$ means convex downward. We calculate $\frac{\partial \bar{v}_{i,j}}{\partial z_{i,j}} = 0$. Then, we derive the optimal sizes of the submitted orders at $t_{i,j}$:

$$\hat{z}^*_{i,j} = \frac{2\hat{B}_{i,j+1} + 2\alpha\sigma^2 - \gamma_1}{2\left(\hat{B}_{i,j+1} + \alpha\sigma^2 + \Lambda_1\right)}X_{i,j}, \qquad j = 1, 2, \ldots, n-1. \tag{85}$$

Substituting (85) into (82), we have

$$\hat{v}_{i,j}(X_{i,j}, b_i, g_{i,j}, h_i) = \hat{B}_{i,j}X_{i,j}^2 + \left(\gamma_1 g_{i,j} + \Lambda_2 b_i + \gamma_2 h_i\right)X_{i,j}, \tag{86}$$

where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n-1$ and

$$\hat{B}_{i,j} = \frac{4(\gamma_1 + \Lambda_1)\hat{B}_{i,j+1} + 4\alpha\gamma_1\sigma^2 + 4\alpha\Lambda_1\sigma^2 - \gamma_1^2}{4\left(\hat{B}_{i,j+1} + \alpha\sigma^2 + \Lambda_1\right)}. \tag{87}$$

We know from (87) that if $\hat{B}_{i,j+1} > \gamma_1/2$, then $\hat{B}_{i,j} > \gamma_1/2$.

Step 3. We derive the optimal liquidation strategy $\hat{z}_{i,n}^*$ and the value function $\hat{v}_{i,n}(X_{i,n}, b_i, g_{i,n}, h_i)$ in the ETC. We suppose that $\hat{v}_{i+1,1}(X_{i+1,1}, b_{i+1}, g_{i+1,1}, h_{i+1}) = \hat{B}_{i+1,1}X_{i+1,1}^2 + (\Lambda_2 b_{i+1} + \gamma_1 g_{i+1,1} + \gamma_2 h_{i+1})X_{i+1,1}$; then, we derive the value function:

$$\begin{aligned}
\hat{v}_{i,n}(X_{i,n}, b_i, g_{i,n}, h_i) &= \inf_{0 \le z_{i,n} \le X_{i,n}} \mathbb{E}_{i,n}\left[\begin{array}{l}\alpha\tilde{\sigma}^2(X_{i,n} - a_{i,n})^2 + (\Lambda_2 b_i + \gamma_1 g_{i,n} + \gamma_2 h_i)a_{i,n} \\ +\hat{v}_{i+1,1}\left(X_{i,n} - a_{i,n}, b_i + z_{i,n} - a_{i,n}, g_{i,n}, h_i + a_{i,n}\right)\end{array}\right] \\
&= \inf_{0 \le z_{i,n} \le X_{i,n}} p\left(\begin{array}{l}\left(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2\right)\left(X_{i,n} - z_{i,n}\right)^2 + \Lambda_2 b_i X_{i,n} + \gamma_1 g_{i,n} X_{i,n} \\ +\gamma_2(h_i + z_{i,n})(X_{i,n} - z_{i,n}) + \gamma_2 h_i z_{i,n}\end{array}\right) \\
&\quad + (1-p)\left((\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2)X_{i,n}^2 + \left(\Lambda_2\left(b_i + z_{i,n}\right) + \gamma_1 g_{i,n} + \gamma_2 h_i\right)X_{i,n}\right) \\
&:= \inf_{0 \le z_{i,n} \le X_{i,n}} \bar{v}_{i,n}(z_{i,n}),
\end{aligned} \tag{88}$$

where $i = 1, 2, \ldots, N-1$. Taking partial derivatives $\bar{v}_{i,n}$ with respect to $z_{i,n}$, we have

$$\frac{\partial \bar{v}_{i,n}}{\partial z_{i,n}} = -2p\left(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2\right)\left(X_{i,n} - z_{i,n}\right) + \Lambda_2(1-p)X_{i,n} + \gamma_2 p\left(X_{i,n} - 2z_{i,n}\right), \tag{89}$$

$$\frac{\partial^2 \bar{v}_{i,n}}{\partial z_{i,n}^2} = 2p\left(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2 - \gamma_2\right). \tag{90}$$

Subsequently, we divide the optimal strategy in the ETC into two cases: (i)$\frac{\partial^2 \bar{v}_{i,n}}{\partial z_{i,n}^2} > 0$; (ii)$\frac{\partial^2 \bar{v}_{i,n}}{\partial z_{i,n}^2} \le 0$.

If $\frac{\partial^2 \bar{v}_{i,n}}{\partial z_{i,n}^2} > 0$, in combination with $0 \le z_{i,n} \le X_{i.n}$, the investor has the optimal size of submitted order at $t_{i,n}$:

$$\hat{z}_{i,n}^* = \begin{cases} X_{i,n}, & \frac{p}{1-p}\gamma_2 \ge \Lambda_2, \\ \frac{p(2\hat{B}_{i+1,1} + 2\alpha\tilde{\sigma}^2 - \gamma_2) - (1-p)\Lambda_2}{2p\left(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2 - \gamma_2\right)}X_{i,n}, & \frac{p}{1-p}(2\hat{B}_{i+1,1} + 2\alpha\tilde{\sigma}^2 - \gamma_2) \ge \Lambda_2 > \frac{p}{1-p}\gamma_2, \\ 0, & \frac{p}{1-p}(2\hat{B}_{i+1,1} + 2\alpha\tilde{\sigma}^2 - \gamma_2) < \Lambda_2, \end{cases} \tag{91}$$

where $i = 1, 2 \ldots N-1$. If $\frac{\partial^2 \bar{v}_{i,n}}{\partial z_{i,n}^2} \le 0$, in combination with $0 \le z_{i,n} \le X_{i.n}$, the investor has the optimal size of submitted order at $t_{i,n}$:

$$\hat{z}_{i,n}^* = \begin{cases} X_{i,n}, & \frac{p}{1-p}(\alpha\tilde{\sigma}^2 + \hat{B}_{i+1,1}) \ge \Lambda_2, \\ 0, & \frac{p}{1-p}(\alpha\tilde{\sigma}^2 + \hat{B}_{i+1,1}) < \Lambda_2, \end{cases} \tag{92}$$

where $i = 1, 2 \ldots N-1$. In conclusion, the investor has the optimal size of submitted orders at $t_{i,n}$:

$$\hat{z}^*_{i,n} = \begin{cases} X_{i,n}, & \bar{\Lambda}_s(i) \geq \Lambda_2, \\ \frac{p(2\hat{B}_{i+1,1}+2\alpha\tilde{\sigma}^2-\gamma_2)-(1-p)\Lambda_2}{2p\left(\hat{B}_{i+1,1}+\alpha\tilde{\sigma}^2-\gamma_2\right)} X_{i,n}, & \bar{\Lambda}_b(i) \geq \Lambda_2 > \bar{\Lambda}_s(i), \\ 0, & \bar{\Lambda}_b(i) < \Lambda_2, \end{cases} \tag{93}$$

where $i = 1, 2 \ldots N-1$, $\bar{\Lambda}_b(i) := \frac{p}{1-p} \max(2\hat{B}_{i+1,1} + 2\alpha\tilde{\sigma}^2 - \gamma_2, \hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2)$ and $\bar{\Lambda}_s(i) := \frac{p}{1-p} \min(\gamma_2, \hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2)$. Substituting (93) into (88), we derive the value function at $t_{i,n}$:

$$\hat{v}_{i,n}(X_{i,n}, b_i, g_{i,n}, h_i) = \hat{B}_{i,n} X^2_{i,n} + \left(\gamma_1 g_{i,n} + \Lambda_2 b_i + \gamma_2 h_i\right) X_{i,n}, \tag{94}$$

where $i = 1, 2, \ldots, N-1$ and

$$\hat{B}_{i,n} = \begin{cases} (1-p)\left(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2 + \Lambda_2\right), & \bar{\Lambda}_s(i) \geq \Lambda_2, \\ \frac{-(\gamma_2 p - \Lambda_2(1-p))^2}{4p\left(\hat{B}_{i+1,1}+\alpha\tilde{\sigma}^2-\gamma_2\right)} + (1-p)(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2 + \Lambda_2), & \bar{\Lambda}_b(i) \geq \Lambda_2 > \bar{\Lambda}_s(i), \\ \hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2, & \bar{\Lambda}_b(i) < \Lambda_2. \end{cases} \tag{95}$$

From (95) and $\gamma_1 < 2(1-p)\alpha\tilde{\sigma}^2/p$, we derive $B_{i,n} > \gamma_1/2$ if $B_{i+1,1} > \gamma_1/2$.

Appendix 5: Some coefficients

Appendix 5.1: Coefficients of Proposition 3.1

$$B_{N,n-1} = \Lambda_1, \tag{96}$$

$$B_{i,j} = \frac{\Lambda_1\left(\alpha\sigma^2 + B_{i,j+1}\right)}{\alpha\sigma^2 + \Lambda_1 + B_{i,j+1}}, \tag{97}$$

where $\{i, j | i \cdot n + j < n \cdot (N+1) - 1, i = 1, 2, \ldots, N, j = 1, 2, \ldots, n-1\}$.

$$B_{i,n} = \begin{cases} \frac{(1-p)\left(4p(B_{i+1,1}+\alpha\tilde{\sigma}^2)^2+4p\Lambda_2(B_{i+1,1}+\alpha\tilde{\sigma}^2)-(1-p)\Lambda_2^2\right)}{4p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}, & \frac{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1, \\ B_{i+1,1} + \alpha\tilde{\sigma}^2, & \frac{2p(B_{i+1,1}+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1, \end{cases} \tag{98}$$

where $i = 1, 2, \ldots, N-1$.

$$\bar{i} = \max\left(\left\{i = 1, 2, \ldots, N-1 \middle| \frac{2p(B_{i+1,1} + \alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1\right\} \cup 0\right). \tag{99}$$

Appendix 5.2: Coefficients of Proposition 4.1

$$B_{1,j} = \frac{c_5^n\left(2\alpha\sigma^2\Lambda_1 + \left(-\alpha\sigma^2 + \sqrt{\alpha\sigma^2(\alpha\sigma^2 + 4\Lambda_1)}\right)B_{1,n}\right) + c_5^j\left(-2\alpha\sigma^2\Lambda_1 + \left(\alpha\sigma^2 + \sqrt{\alpha\sigma^2(\alpha\sigma^2 + 4\Lambda_1)}\right)B_{1,n}\right)}{c_5^j\left(-\alpha\sigma^2 + \sqrt{\alpha\sigma^2(\alpha\sigma^2 + 4\Lambda_1)} - 2B_{1,n}\right) + c_5^n\left(\alpha\sigma^2 + \sqrt{\alpha\sigma^2(\alpha\sigma^2 + 4\Lambda_1)} + 2B_{1,n}\right)}, \tag{100}$$

$$B_{1,n} = \begin{cases} \frac{(1-p)}{4p(c_2+\alpha\tilde{\sigma}^2)}\left(4pc_2^2 - (1-p)\Lambda_2^2 + 4p\alpha\tilde{\sigma}^2\Lambda_2 + 4p\alpha^2\tilde{\sigma}^4 + 4pc_2(\Lambda_2 + 2\alpha\tilde{\sigma}^2)\right), & \frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} > 1, \\ c_2 + \alpha\tilde{\sigma}^2, & \frac{2p(c_2+\alpha\tilde{\sigma}^2)}{(1-p)\Lambda_2} \leq 1, \end{cases} \tag{101}$$

$$c_1 = \operatorname{arcosh}\left(\frac{\alpha\sigma^2}{2\Lambda_1} + 1\right), \tag{102}$$

$$c_2 = \operatorname{csch}^2((n-1)c_1)\sinh(\frac{c_1}{2})\left(\sinh(\frac{c_1}{2}) - \sinh(\frac{1}{2}(5-4n)c_1)\right)\Lambda_1, \tag{103}$$

$$c_5 = 1 + \frac{\alpha\sigma^2\left(\alpha\sigma^2 + \sqrt{\alpha\sigma^2(4\Lambda_1 + \alpha\sigma^2)}\right)}{2\Lambda_1^2} + \frac{2\alpha\sigma^2 + \sqrt{\alpha\sigma^2(4\Lambda_1 + \alpha\sigma^2)}}{\Lambda_1}, \tag{104}$$

where $j = 1, 2, \ldots, n$.

Appendix 5.3: Coefficients of Proposition 5.1

$$\hat{B}_{N,n-1} = \Lambda_1 + \gamma_1, \tag{105}$$

$$\hat{B}_{i,j} = \frac{4(\gamma_1 + \Lambda_1)\hat{B}_{i,j+1} + 4\alpha\gamma_1\sigma^2 + 4\alpha\Lambda_1\sigma^2 - \gamma_1^2}{4\left(\hat{B}_{i,j+1} + \alpha\sigma^2 + \Lambda_1\right)}, \tag{106}$$

where $\{i, j | i \cdot n + j < n \cdot (N+1) - 1, i = 1, 2, \ldots, N, j = 1, 2, \ldots, n-1\}$.

$$\hat{B}_{i,n} = \begin{cases} (1-p)\left(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2 + \Lambda_2\right), & \bar{\Lambda}_s(i) \geq \Lambda_2, \\ \frac{-(\gamma_2 p - \Lambda_2(1-p))^2}{4p\left(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2 - \gamma_2\right)} + (1-p)(\hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2 + \Lambda_2), & \bar{\Lambda}_b(i) \geq \Lambda_2 > \bar{\Lambda}_s(i), \\ \hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2, & \bar{\Lambda}_b(i) < \Lambda_2, \end{cases} \tag{107}$$

$$\bar{\Lambda}_b(i) = \frac{p}{1-p}\max(2\hat{B}_{i+1,1} + 2\alpha\tilde{\sigma}^2 - \gamma_2, \hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2), \tag{108}$$

$$\bar{\Lambda}_s(i) = \frac{p}{1-p}\min(\gamma_2, \hat{B}_{i+1,1} + \alpha\tilde{\sigma}^2), \tag{109}$$

where $i = 1, 2, \ldots, N-1$.

Appendix 5.4: The optimal liquidation strategy without the ETC

When the ETC does not exist, the optimal sizes of submitted orders in the RTS are

$$\bar{z}_{i,j}^* = \begin{cases} \frac{\alpha\sigma^2 + \bar{B}_{i,j+1}}{\alpha\sigma^2 + \Lambda_1 + \bar{B}_{i,j+1}}\bar{X}_{i,j}, & i \neq N || j \neq n-1, \\ \bar{X}_{N,n-1}, & i = N \&\& j = n-1, \end{cases} \tag{110}$$

where $i = 1, 2 \ldots N$, $j = 1, 2, \ldots, n-1$ and

$$\bar{B}_{N,n-1} = \Lambda_1, \tag{111}$$

$$\bar{B}_{i,j} = \frac{\Lambda_1\left(\alpha\sigma^2 + \bar{B}_{i,j+1}\right)}{\alpha\sigma^2 + \Lambda_1 + \bar{B}_{i,j+1}}, \tag{112}$$

$$\bar{B}_{i,n} = \bar{B}_{i+1,1} + \alpha\tilde{\sigma}^2. \tag{113}$$

Appendix 6: Rules on trading in the extended trading close

The Nasdaq introduced the ETC on March 7, 2022, and the rules on trading in the ETC of the Nasdaq are as follows:

- Trading Hours: 16:00:05–16:05:00 (the closing time is 16:00);
- Price: the Nasdaq closing cross price;
- Matching Rules: the principle of time priority;
- Disseminated Information: imbalance side (the side of the imbalance), imbalance shares (the number of unmatched shares in the ETC), paired shares (the aggregate number of shares executed in the ETC) and the Nasdaq closing cross price.

The ETC is called the After-Hours Fixed-Price in the SSE STAR, SZSE ChiNext and TWSE. The SSE STAR introduced the After-Hours Fixed-Price on July 22, 2019. The rules on trading in the After-Hours Fixed-Price of the SSE STAR are as follows:

- Trading Hours: 15:05-15:30 (the closing time is 15:00);
- Price: the closing price of the day of trading;
- Matching Rules: the principle of time priority;
- Disseminated Information: the real-time buy or sell quantities, daily accumulated trading volume and the value of stocks during After-Hours Fixed-Price trading and the closing price.

The SZSE ChiNext introduced the After-Hours Fixed-Price on August 24, 2020. The rules on trading in the After-Hours Fixed-Price of the SZSE ChiNext:

- Trading Hours: 15:05-15:30 (the closing time is 15:00);
- Price: the closing price of the day of trading;
- Matching Rules: the principle of time priority;
- Disseminated Information: the volume of unexecuted real-time buy or sell orders, the accumulated turnover volume and the turnover value of After-Hours Fixed-Price orders and the closing price.

The TWSE introduced the After-Hours Fixed-Price on March 8, 2000. The rules on trading in the After-Hours Fixed-Price of the TWSE are as follows:

- Trading Hours: 14:00-14:30 (the closing time is 13:30);
- Price: the closing price of the security on the date;
- Matching Rules: computer-determined random;
- Disseminated Information: the volume of trades executed and the volume of trading quotes.

Abbreviations

ETC	Extended trading close
RTS	Regular trading session
Nasdaq	National association of securities dealers automated quotations
SSE STAR	Science and technology innovation board of shanghai stock exchang
SZSE ChiNext	ChiNext market of shenzhen stock exchange
TWSE	Taiwan stock exchange

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Competing interests
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