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Return direction forecasting: a conditional autoregressive shape model with beta density

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Abstract

This paper derives a new decomposition of stock returns using price extremes and proposes a conditional autoregressive shape (CARS) model with beta density to predict the direction of stock returns. The CARS model is continuously valued, which makes it different from binary classification models. An empirical study is performed on the US stock market, and the results show that the predicting power of the CARS model is not only statistically significant but also economically valuable. We also compare the CARS model with the probit model, and the results demonstrate that the proposed CARS model outperforms the probit model for return direction forecasting. The CARS model provides a new framework for return direction forecasting.

Keywords: Return direction forecasting, Price extremes, CARS, Beta distribution

Introduction

The predictability of the stock market has long been an interesting and important topic in both financial economics and econometrics. This topic is interesting from different perspectives. For practitioners, better forecasts mean better investment returns. For theoretical researchers, market predictability is highly related to asset pricing theories. There is a vast volume of literature that focuses on the prediction of stock returns using public information. For a comprehensive and up-to-date overview of the literature on stock return predictability, we refer to Rapach and Zhou (2013).

It is well-known that the overall level, or the conditional mean of the excess return, is notoriously difficult to forecast (Welch and Goyal 2008); however, the direction of stock returns is much more predictable. For example, Christoffersen and Diebold (2006) and Christoffersen et al. (2007) established the theoretical connection between asset return volatility and return direction predictability and demonstrated that return direction is predictable. Many studies have documented that only the direction of asset returns is predictable (see, among others, Breen et al. 1988; Hong and Chung 2006; Chung and Hong 2007).

Directional predictability is important for market timing, which is crucial for asset allocation decisions between stock and risk-free interest rate investments. Leitch and Tanner (1991, 1995) show that the direction-of-change criterion may be better able to capture a utility-based measure of forecasting performance such as economic profits

(see Granger and Pesaran 2000; Pesaran and Skouras 2001 for further discussion). Various classification-based qualitative models, such as traditional static logit and probit models were considered by Leung et al. (2000), whereas Hong and Chung (2006), Rydberg and Shephard (2003), and Anatolyev and Gospodinov (2007) used the so-called autologistic model, while Nyberg (2011) used the so-called dynamic probit model to predict the return direction. Nyberg and Ponka (2016) extended the dynamic probit model to a bivariate case and include interaction effects with the United States for return predictions in other financial markets. The recent academic literature shows a rising interest in using machine learning to forecast return direction (Kara et al. 2011; Qiu and Song 2016; Zhong and Enke 2019).

This study attempts to provide a new approach to forecast the direction of stock return. Traditionally, return direction forecasting models are based on the sign of return, which totally ignores the possible value of price extremes in return direction forecasting. The recent academic literature shows that price extremes are informative for return forecasting (see Xie and Wang 2013, 2018). Thus, the question of "how to use price extremes to forecast the direction of stock returns" is of great interest and also the focus of this paper.

With price extremes, asset return can be decomposed into a linear combination of two parts, and the direction of the stock return is determined by the relative strength of these two parts. A simple transformation shows that direction prediction is equivalent to a ratio prediction. To model the dynamics of this ratio series, a conditional autoregressive shape model with beta (henceforth B-CARS) density is proposed. The specification of the B-CARS model is much like the GARCH model and continuously valued, which makes it totally different from classification-based qualitative models.

An empirical study is performed on the monthly US stock index to evaluate the forecasting power of the B-CARS model in return direction forecasting. The results show that the return direction is significantly predictable in both statistical and economic senses, judging by either the Sharpe ratio or utility gain. According to the utility gain, investors would like to pay an annual return of 3.84% to access the forecasts relative to the simple market portfolio, and such payment is even as much as 21.70% in market recession.

We also compare the B-CARS model with the commonly used classification probit model. The results show that the B-CARS model significantly outperforms the probit model in an economic sense. Another interesting finding is that the macroeconomic and financial variables selected by the B-CARS model are different from those by the probit model, which confirms that the B-CARS model is different from the probit model in a new perspective.

The main contributions of this paper are presented as follows: first, return direction forecasting models are based on the sign of the return, which ignores the possible value of price extremes in return direction forecasting; thus, this paper proposes a method to predict the direction of stock return by using stock extreme value information. Second, we conduct empirical analysis on the monthly US stock index to evaluate the forecasting power of the B-CARS model in return direction forecasting and compare the B-CARS model with the commonly used classification probit model. The remainder of this paper is organized as follows: "Econometric methods" section introduces the B-CARS model. "Empirical results on

S &P500" section empirically investigates the forecasting power of the B-CARS model on the US stock market index. "Further evidence" section compares the B-CARS model with the classic classification probit model. We conclude in "Conclusion" section.

Econometric methods

In this section, we will first show that return can be decomposed into a multiplication of two components using price extremes and then propose a new econometric method for return direction forecasting.

Return decomposition

Let p_t be the logarithmic price of a speculative asset observed at time t. The high price h_t over time interval [t-1, t] is defined as

$$h_t = \max_{t-1 \le \tau \le t} p_\tau \tag{1}$$

The asset return r_t over [t-1, t] can thus be expressed as

$$r_t = p_t - p_{t-1} = \underbrace{(h_t - p_{t-1})}_{u_t} - \underbrace{(h_t - p_t)}_{d_t} \tag{2}$$

where u_t measures the quantity of maximum price increase, and d_t measures the quantity of maximum price decrease.

Let R_t=u_t+d_t, which measures the total price variation over [t-1, t]. We define

$$ur_t = \frac{u_t}{R_t},\ dr_t = \frac{d_t}{R_t} \tag{3}$$

where ur_t (dr_t) gauges the contribution of maximum price increase (decrease) to the total price variation. We call ur_t the up ratio and dr_t the down ratio. It is evident that

$$ur_t \ge 0,\ dr_t \ge 0 \text{ and } ur_t + dr_t = 1. \tag{4}$$

From Eqs. (2)–(4), it follows that asset return can be rewritten as

$$r_t = R_t(ur_t - dr_t) = R_t(2ur_t - 1) \tag{5}$$

which means that the return can be decomposed into a multiplication of a return variation and a ratio. Given that R_t>0, Eq. (5) implies that if $ur_t > \frac{1}{2}$ then the return is positive. This fact hints that we can predict the direction of the return by predicting ur_t: if ur_t is predicted to be larger than 1/2, then we should predict a positive return; otherwise, we should forecast a negative return.

The model

Given that $ur_t \in$[0, 1] and is continuously valued, a natural choice for its density is the beta distribution

$$ur_t \sim \text{Beta}(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ \alpha,\ \beta > 0 \tag{6}$$

where $\Gamma(.)$ is a gamma function. The shape of the beta density is controlled by the ratio of α over β: if $\frac{\alpha}{\beta}$=1, the beta density is symmetric around 1/2; if $\frac{\alpha}{\beta}$>1, the density function is left-skewed about 1/2; otherwise, it is right-skewed about 1/2.

Given the fact that the expectation of ur_t is

$$E(ur_t) = \frac{\alpha}{\alpha + \beta}, \tag{7}$$

we thus propose the following model to describe the dynamics of ur_t

$$ur_t \sim Beta(x, \alpha_t, \beta),\ k_t = \omega + \sum_{i=1}^{P} \gamma_i k_{t-i} + \sum_{j=1}^{Q} \tau_j ur_{t-j} \tag{8}$$

where x is a variable in the probability density function, k_t=$\alpha_t/(\alpha_t + \beta)$, and

$$\omega > 0,\ \gamma_i,\ \tau_j \geq 0,\ \omega + \sum_{i=1}^{P} \gamma_i + \sum_{j=1}^{Q} \tau_j \leq 1 \tag{9}$$

As k_t controls the shape, Eq. (8) is thus called the conditional autoregressive shape of order (P, Q) with beta density, or B-CARS(P,Q) for short. Condition (9) is required to ensure that $k_t \in [0, 1]$.

Let information filtration Ω_t={k_t, k_{t-1},...}. The conditional expectation of ur_t is given by

$$E[ur_t|\Omega_t] = k_t \tag{10}$$

The unconditional expectation of ur_t is

$$E[ur_t] = \frac{\omega}{1 - \sum_{i=1}^{P} \gamma_i - \sum_{j=1}^{Q} \tau_j} = E[k_t] \tag{11}$$

The B-CARS model can be further extended by including exogenous variables,

$$ur_t \sim Beta(x, \alpha_t, \beta),\ k_t = \omega + \sum_{i=1}^{P} \gamma_i k_{t-i} + \sum_{j=1}^{Q} \tau_j ur_{t-j} + \kappa X_{t-1} \tag{12}$$

where X_{t-1} is an exogeneous variable. To ensure that $k_t \in [0, 1]$, we also require that $X_t \in [0, 1]$ and that

$$\omega > 0,\ \gamma_i,\ \tau_j,\ \kappa \geq 0,\ \omega + \sum_{i=1}^{P} \gamma_i + \sum_{j=1}^{Q} \tau_j + \kappa \leq 1 \tag{13}$$

Model estimation

The maximum likelihood estimation (MLE) method can be used to estimate the unknown parameters in the B-CARS model.

Conditional on information filtration Ω_T, the joint density of $(ur_T, ur_{T-1}, \ldots, ur_1)$ is given by

$$f(ur_T, \ldots, ur_1|\Omega_T; \beta) = f_T(ur_T|k_T; \beta) \ldots f_1(ur_1|k_1; \beta) \tag{14}$$

where

$$f_t(ur_t|\Omega_t; \beta) = \frac{\Gamma(\alpha_t + \beta)}{\Gamma(\alpha_t)\Gamma(\beta)} ur_t^{\alpha_t - 1}(1 - ur_t)^{\beta - 1} \tag{15}$$

The log-likelihood function of the joint density is given by

$$\text{llf} = \sum_{t=1}^{T} \left[ln \frac{\Gamma(\alpha_t + \beta)}{\Gamma(\alpha_t)\Gamma(\beta)} + (\alpha_t - 1) ln(ur_t) + (\beta - 1) ln(1 - ur_t) \right] \tag{16}$$

By maximizing the likelihood function, the unknown parameters in the B-CARS model can be obtained.

Empirical results on S &P500

This section is used to evaluate the performance of the B-CARS model for forecasting the direction of stock returns.

Data

We collect the monthly Standard & Poor's 500 (S &P500) stock index data for the sample period from January 1928 to December 2019 with 1,104 data observations. For each month, four pieces of price information—open, high, low, and close—are reported.[1] The data set is downloaded from the finance subdirectory of the website "www.finance.yahoo.com".

Table 1 presents the summary statistics of monthly return and up ratio. Consistent with the well-documented facts, the return on S &P500 is far from being normal with high kurtosis and negative skewness. Both the mean and median of the up ratio are larger than 0.5, indicating that asset returns are more likely to be positive. Figure 1 presents the histogram and autocorrelation plots of the return and up ratio. The left panel shows the sample autocorrelations of return series and its histogram, and the right panel presents the sample autocorrelations of the up ratio series and its histogram. Two interesting findings emerge from Fig. 1. First, there is no autocorrelation in the return series, while the autocorrelation is significant in the up ratio series. The sample autocorrelation function (acf) of the up ratio at lag 5 is reported to be 0.123. Second, the distribution of the return is bell-shaped while the distribution of the up ratio is U-like.

We also collect macroeconomic and financial variables, including *risk-free rate (rfree), dividend-price ratio (dp), dividend yield (dy), earning-price ratio (ep), dividend-payout ratio (de), stock variance (svar), book-to-market ratio (bm), net equity expansion (ntis), treasury bill rate (tbl), long-term return (ltr), term spread (tms), default yield spread (dfy), and inflation (infl)*.[2]

[1] As the monthly high price collected from the website does not consider the last monthly close price, the following adjustment h_t^a=max{h_t, c_{t-1}} is used as the true high price in month t.

[2] These macro- and financial variables are downloaded from the homepage of Amit Goyal at www.hec.unil.ch/agoyal/.

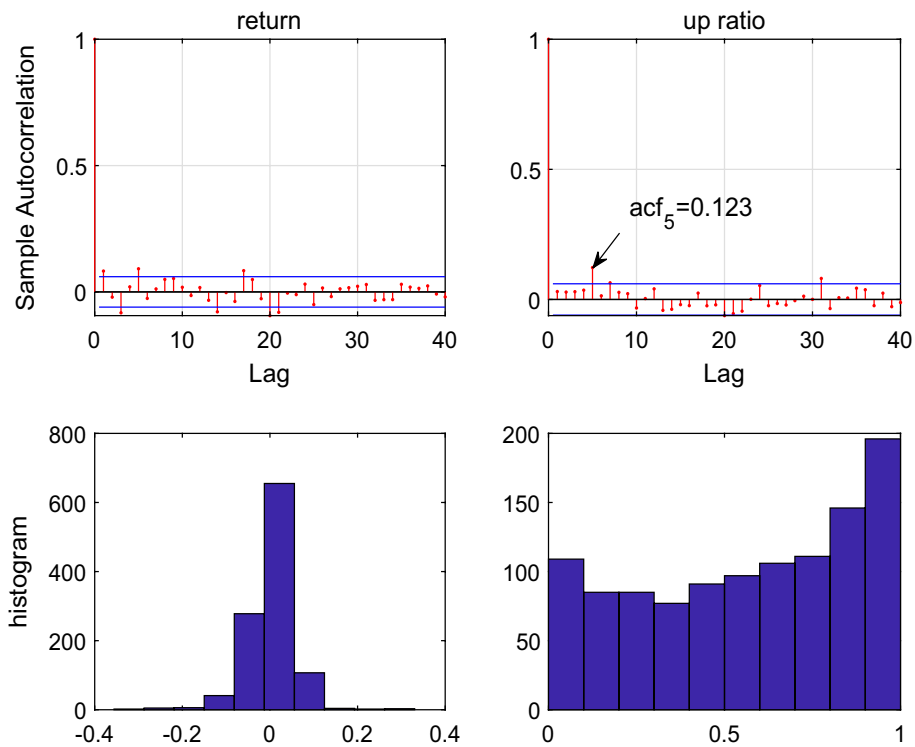

Fig. 1 Histogram and autocorrelation plots of return and up ratio

Table 1 Summary statistics of return and up ratio

Variable	Mean	SD	Median	Min	Max	Skew.	Kurt.
ur	.566	.313	0.609	0	1	−.271	1.820
ret	.005	.054	0.009	−.356	.330	−.623	10.490

- *dividend-price ratio, dp*: difference between the log of dividends paid on the S &P500 index and log of stock prices (S &P500 index), where dividends are measured using a one-year moving sum.
- *dividend yield, dy*: difference between the log of dividends and log of lagged stock prices.
- *earnings-price ratio, ep*: difference between the log of earnings on the S &P 500 index and log of stock prices, where earnings are measured using a one-year moving sum.
- *dividend payout ratio, de*: difference between the log of dividends and log of lagged stock prices.
- *stock variance, svar*: sum of squared daily returns on the S &P 500 index.
- *book-to-market ratio, bm*: ratio of book value to market value for the Dow Jones Industrial Average.
- *net equity expansion, ntis*: ratio of twelve-month moving sums of net issues by NYSE-listed stocks to total end-of-year market capitalization of NYSE stocks.
- *treasure bill rate, tbl*: interest rate on a three-month Treasury bill (secondary market).

Table 2 Descriptive statistics of regularized variables

Variable	Mean	SD	Median	Skew.	Kurt.	Corr
bm	.232	.141	.220	.770	4.349	−0.117
tbl	.207	.190	.176	1.098	4.315	−0.035
ntis	.313	.109	.316	1.666	11.619	−0.069
infl	.290	.066	.289	1.177	17.570	−0.043
ltr	.443	.093	.436	.590	7.385	0.073
svar	.039	.081	.017	5.803	46.792	−0.195
tms	.676	.149	.684	−.449	3.568	−0.026
dfy	.151	.130	.109	2.492	11.938	−0.074
dp	.428	.176	.437	−.152	2.556	−0.099
dy	.438	.177	.449	−.178	2.536	−0.007
ep	.682	.136	.663	−.541	5.473	−0.073
de	.229	.125	.232	1.560	9.095	−0.047

- *long-term return, ltr*: return on long-term government bonds.
- *term spread, tms*: difference between the long-term yield and Treasury bill rate.
- *default yield spread, dfy*: difference between BAA- and AAA-rated corporate bond yields.
- *inflation, infl*: inflation is the Consumer Price Index (All Urban Consumers).

Table 2 presents the summary statistics. All these variables are regularized using the following formula:

$$r(x_t) = \frac{x_t - min\{x_t\}}{max\{x_t\} - min\{x_t\}} \in [0, 1] \tag{17}$$

Regularization is employed to ensure that $r(x_t)$ satisfies the condition of Eq. (12). The last column shows the correlation between the up ratio and these regularized variables.

In-sample estimation

It has been well documented that order (1,1) can well capture the dynamics of financial time series; thus, we only consider the following B-CARS model of order (1,1)[3]

$$k_t = \omega + \gamma_1 k_{t-1} + \tau_1 ur_{t-1} \tag{18}$$

The estimates of the B-CARS(1,1) models are presented in Table 3. Column 1 presents the estimate of the benchmark B-CARS(1,1) model. We find that the up ratio is indeed predictable as its conditional mean, while k_t is reported to be persistent although the R^2 = 0.66% is small. Figure 2 presents the series ur_t and its forecast. It is clear from Fig. 2 that the conditional mean of ur_t is mean-reverting and predictable to some extent.

As it has been presented in "The model" section that the shape of the beta density is controlled by the relative strength of α_t to β, Fig. 3 presents the plots of the α_t and β estimates. Series α_t is calculated using the following formula:

[3] We also estimate CARS(1,2) and CARS(2,1), and the results show that CARS(1,1) is sufficient to capture the up ratio dynamics. We do not present the estimation results in order to save space.

Table 3 Estimates of B-CARS(1,1) model

	B-CARS(1,1)	bm	tbl	−ntis	−infl	ltr	−svar	tms	−dfy	dp	dy	ep	de
ω	0.108 (0.041)	0.106 (0.042)	0.108 (0.042)	0.060 (0.031)	0.108 (0.045)	0.039 (0.046)	0.071 (0.044)	0.104 (0.045)	0.106 (0.041)	0.108 (0.041)	0.108 (0.041)	0.103 (0.040)	0.108 (0.043)
γ_1	0.766 (0.077)	0.767 (0.078)	0.766 (0.086)	0.801 (0.076)	0.766 (0.080)	0.797 (0.075)	0.752 (0.087)	0.769 (0.080)	0.756 (0.085)	0.766 (0.079)	0.766 (0.079)	0.766 (0.077)	0.766 (0.088)
τ_1	0.048 (0.018)	0.048 (0.018)	0.048 (0.018)	0.040 (0.018)	0.048 (0.018)	0.041 (0.018)	0.042 (0.018)	0.047 (0.018)	0.046 (0.018)	0.048 (0.018)	0.048 (0.018)	0.048 (0.018)	0.048 (0.019)
β	0.532 (0.020)	0.532 (0.020)	0.532 (0.020)	0.534 (0.020)	0.532 (0.020)	0.534 (0.020)	0.532 (0.020)	0.532 (0.020)	0.532 (0.020)	0.532 (0.020)	0.532 (0.020)	0.532 (0.020)	0.532 (0.020)
X		0.002 (0.015)	0.000 (0.012)	0.046 (0.020)	0.000 (0.044)	**0.124** (0.054)	0.049 (0.041)	0.002 (0.015)	0.009 (0.018)	0.000 (0.012)	0.000 (0.012)	0.006 (0.015)	0.000 (0.019)
R^2(%)	**0.66**	0.66	0.66	**1.05**	0.66	**1.18**	0.70	0.67	0.70	0.66	0.66	0.72	0.66

Bold represents a better performance

[1] MLE method cannot be applied to the B-CARS model when ur_t=0 or 1. We deal with this problem using the following principle: $ur_t = \begin{cases} \max_t\{ur_t/[1]\} & \text{if } ur_t = 1 \\ \min_t\{ur_t/[0]\} & \text{if } ur_t = 0 \end{cases}$ where $\{ur_t/[v]\}$ means that the elements in ur_t series have been removed if ur_t=v

[2] Numbers in the parentheses are standard errors

[3] As we do not know whether $r(x_t)$ is positively or negatively correlated with k_t, both $r(x_t)$ and 1-$r(x_t)$ are used as exogeneous variables when performing MLE and the one with a higher R^2 is selected to be the final one. Symbol -x means that x is regularized by 1-$r(x_t)$

[4] When performing MLE, the initial five k_ts are set to be the unconditional mean of ur_t

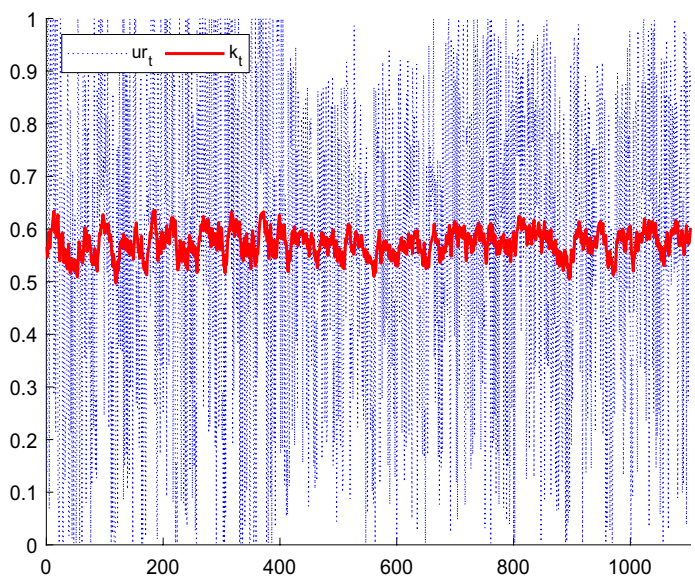

Fig. 2 Time series plots of ur_t and its forecasts

$$\alpha_t = k_t\beta/(1-k_t).$$

Two main findings emerge from Fig. 3. First, both α_t and β are less than 1, indicating that the density of ur_t concentrates more on tail than on central. Second, α_t is larger than β most of the time, indicating that the right tail in the density of ur_t is higher than the left tail. Summarizing these two facts, the density of ur_t is more likely to be J-shaped.

Notably, B-CARS models with exogeneous variables are also estimated, and the results are shown in the remaining columns. Only two variables, *ntis* and *ltr* are found to be significantly important for forecasting ur_t. Different from the findings of

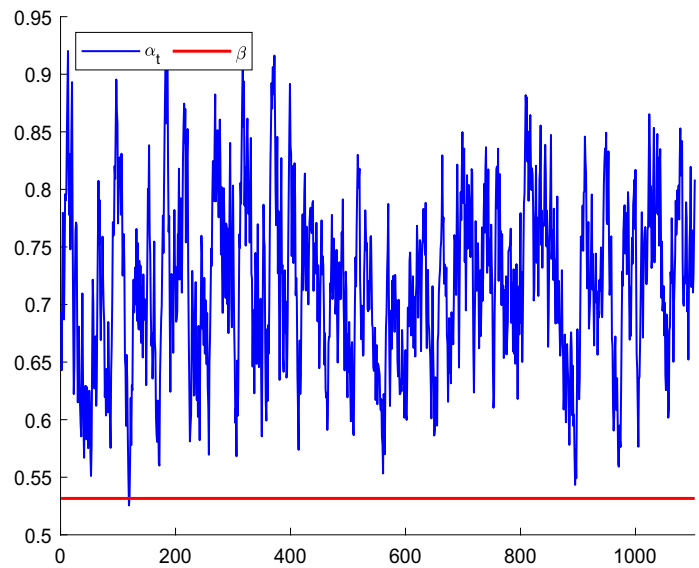

Fig. 3 Time series plots of α_t and β

Christoffersen et al. (2007), market variance is found to be insignificant, and the contribution of *svar* to up ratio forecasting is very small as the R^2 statistic only increases by 0.04% from 0.66% to 0.70%. The reason may be that volatility clustering effect has been removed from return since

$$ur_t = \frac{1}{2}\left(\frac{r_t}{R_t} + 1\right) \tag{19}$$

which indicates that the return series r_t has been standardized by price variation R_t.

Out-of-sample forecast

In-sample fitting does not guarantee out-of-sample predictability. This section investigates the out-of-sample forecasting power of the B-CARS model. For out-of-sample forecasting, an extending window forecasting procedure is used:

Step 1: The first q data observations are used to estimate the model;

Step 2: Using the estimates to forecast k_{q+1};

Step 3: Let q=q+1, and return to step 1. Repeat the above-stated procedures till the end.

Following Campbell and Thompson (2008), the following out-of-sample R-square statistic, R^2_{oos}, is used to evaluate the empirical performance of the B-CARS model relative to the simple historical mean model:

$$R^2_{oos} = 1 - \frac{\sum_{t=q+1}^{T}(ur_t - \hat{ur}_t)^2}{\sum_{t=q+1}^{T}(ur_t - \bar{ur}_t)^2} \tag{20}$$

where $\hat{ur}_t$ are forecasts reported by B-CARS and $\bar{ur}_t$ are historical mean forecasts defined as follows:

Table 4 Out-of-sample *R*-square statistics of different models

R^2_{oos}(%)	Whole sample	Expansion	Recession
B-CARS(1,1)	0.099	0.056	0.329**
*ntis*_	−0.044	0.231**	−0.160
*ltr*_	0.164**	1.11**	0.557*

[1] We use *x*_ to represent B-CARS(1,1) with *x* as exogeneous variable

[2] We use ***,** and * to represent significance at the level of 1%, 5% and 10% respectively

$$\bar{ur}_t = \frac{1}{t-1}\sum_{k=1}^{t-1} ur_k,\ t = q+1,\ q+2, \ldots,\ T$$

Historical mean model simulates efficient market forecasting. Furthermore, B-CARS beats the historical mean if R^2_{oos} is positive, otherwise the historical mean model outperforms.

Herein, the first 500 observations are used as the initial window, and the remaining 603 observations are used as testing samples. Table 4 presents the out-of-sample *R*-square statistics. We only report variables that are found to be important in the in-sample estimation. Two findings can be obtained from Table 4. First, the positive R^2_{oos}=0.099% indicates that the benchmark B-CARS(1,1) model outperforms the simple historical mean model, which is consistent with the in-sample results. Second, including exogeneous variables does not necessarily improve out-of-sample forecasting performance. For example, when *ntis* is included, the R^2_{oos} decreases from 1.05% to −0.044%.

Welch and Goyal (2008) present comprehensive evidence that few models significantly beat the historical mean forecast; therefore, a null hypothesis of interest to investors is $R^2_{oos} \leq 0$ against the alternative hypothesis that $R^2_{oos} > 0$. We test this hypothesis by using the mean square forecast error (MSFE)-adjusted statistic of Clark and West (2007). Define

$$f_t = (ur_t - \bar{ur}_t)^2 - [(ur_t - \hat{ur}_t)^2 - (\bar{ur}_t - \hat{ur}_t)^2]. \tag{21}$$

Then, the Clark and West (2007) MSFE-adjusted statistic is the *t*-statistic from the regression of f_t on a constant. The results are also presented in Table 4. It is found only the B-CARS model with *ltr* significantly outperforms the historical mean, indicating that variable *ltr* is informative to return direction forecasting both in-sample and out-of-sample.

It has been documented the predictability of stock returns is related to business cycle (see Rapach et al. 2010); thus, we also calculate the out-of-sample MSFE-adjusted statistics over business expansion and business recession. The expansion and recession time periods are dated by the NBER. The results show that the B-CARS model with *ltr* outperforms the historical mean model both in economic recession and expansion. The benchmark B-CARS model outperforms only in economic recession, while the B-CARS model with *ntis* dominates only in economic expansion.

Table 5 Summary statistics of returns on different model based switching strategies

	Mean	Std.	Median	Max	Min	Skew.	SR(%)	Δ(U)
Whole sample								
Market	5.882E–03	0.044	9.381E–03	0.151	−0.245	−0.721	4.56	
B-CARS(1,1)	6.410E–03	0.034	5.375E–03	0.124	−0.245	−1.051	7.44	2.00
*ntis_*	6.629E–03	0.038	6.217E–03	0.124	−0.245	−1.007	7.17	1.70
*ltr_*	7.925E–03	0.034	6.033E–03	0.124	−0.158	−0.286	**11.96**	**3.84**
Expansion								
Market	7.80E–03	0.039	10.432E–03	0.124	−0.245	−0.824	10.52	
B-CARS(1,1)	6.287E–03	0.035	5.138E–03	0.124	−0.245	−1.029	7.61	−1.18
*ntis_*	7.838E–03	0.036	6.108E–03	0.124	−0.245	−0.921	11.64	2.84
*ltr_*	7.959E–03	0.033	6.025E–03	0.124	−0.158	−0.353	**13.06**	**2.99**
Recession								
Market	−6.133E–03	0.063	−5.978E–03	0.151	−0.186	−0.128	−18.23	
B-CARS(1,1)	7.181E–03	0.028	5.958E–03	0.105	−0.107	−1.188	**6.29**	**21.70**
*ntis_*	−0.948E–03	0.050	6.292E–03	0.105	−0.186	−0.933	−12.78	8.92
*ltr_*	7.713E–03	0.038	6.133E–03	0.116	−0.099	−0.006	6.07	21.12

Bold represents a better performance

[1] We use *x_* to represent B-CARS(1,1) with *x* as exogeneous variable

[2] We use Δ(U) to represent utility gain

Economic value

The results in "In-sample estimation" and "Out-of-sample forecast" sections confirm that the up ratio is predictable both in-sample and out-of-sample and that this predictability is significant in a statistical sense. This section investigates if this predictability can be translated into economic value. This is important, as for financial analysts and practitioners, the most important model evaluation criterion is the return on their investment.

In this paper, a switching trading strategy is employed. At the beginning of each month, the investors make an asset allocation decision. They can shift their assets either into stocks or the risk-free Treasury bills and hold either of these alternatives till the next decision date. The switching trading strategy is presented as

$$r_t^s = r_t I_t + (1 - I_t) r_t^f \tag{22}$$

where r_t is the return on stock index, r_t^f is the risk-free return, r_t^s is the return on the switching strategy, and

$$I_t = \begin{cases} \text{Stock} & \hat{ur}_t > \bar{ur}_t \\ \text{Treasury Bills} & \hat{ur}_t \leq \bar{ur}_t \end{cases}$$

At time $t-1$, if the forecast of the B-CARS model $\hat{ur}_t$ is larger than historical mean forecast $\bar{ur}_t$, investors hold the stock index, otherwise they hold the risk-free asset at time t.

Table 5 reports the summary statistics of the returns on different model-based switching strategies together with the commonly used Sharpe ratio (SR)

$$SR(x) = \frac{\mu_x}{\sigma_x} \tag{23}$$

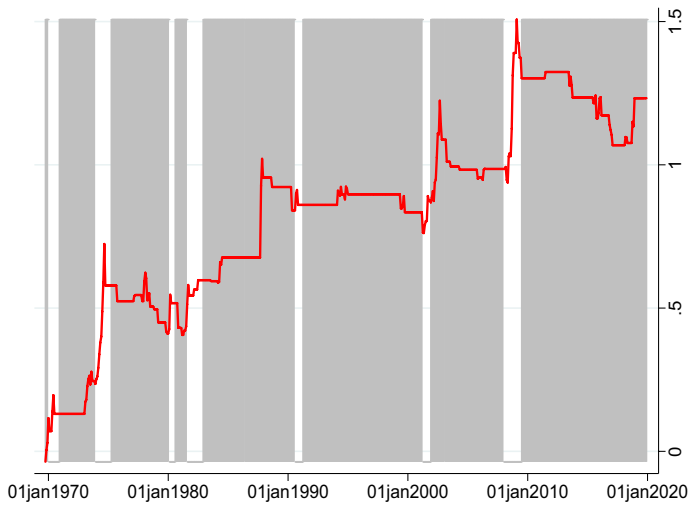

Fig. 4 Time series plots of cumulative returns from different model based switching strategies

where μ_x is the mean excess return based on model x and σ_x is the sample standard deviation of the mean excess return. Compared with the market portfolio, all the switching trading strategies deliver higher mean returns and smaller standard deviations. The Sharpe ratio statistics show that our switching trading strategies outperform the market portfolio. The Sharpe ratio of the *ltr*_based trading strategy is even two times larger than that of the market portfolio.

We also report in Table 5, the summary statistics of returns on different trading strategies over a business cycle. Consistent with the out-of-sample R^2 statistics, the Sharpe ratio statistics show that the *ltr*_based trading strategy dominates the market portfolio in both expansion and recession, and the benchmark B-CARS-based trading strategy outperforms only in market recession. Interestingly, we find that the *ntis*_based trading strategy outperforms the market portfolio in both recession and expansion, which is different from the results of the out-of-sample R^2 statistics.

Figure 4 presents the cumulative return difference between the *ltr*_based trading strategy and the market portfolio. It can be observed from Fig. 4 that the outperformance of the *ltr*_based trading strategy mainly occurs when the market is in recession.

Following Campbell and Thompson (2008) and Rapach et al. (2010), we further calculate the utility gain of the switching strategy relative to the simple market portfolio. Over the out-of-sample period, the market portfolio investors realize an average utility level of

$$\hat{v}_0 = \hat{u}_0 - 0.5\gamma\hat{\sigma}_0^2 \tag{24}$$

where $\hat{u}_0$ and $\hat{\sigma}_0^2$ are the sample mean and variance, respectively, over the out-of-sample period for the return on the market portfolio. Letter γ is the relative risk aversion parameter. If the same investor uses the switching strategy, then he/she realizes an average utility level of

$$\hat{v}_s = \hat{u}_s - 0.5\gamma\hat{\sigma}_s^2 \tag{25}$$

where $\hat{u}_s$ and $\hat{\sigma}_s^2$ are, respectively, the sample mean and variance of the return on the switching strategy over the out-of-sample period. The utility gain is measured as the difference between (25) and (24). We multiply this difference by 1200 to express it in average annualized percentage return. The utility gain can be interpreted as the portfolio management fee that an investor would be willing to pay to have access to the switching strategy.

The utility gains are reported in the last column in Table 5. We report results for $\gamma = 3$; the results are qualitatively similar for other reasonable γ values. In general, the results confirm that model-based trading strategies can improve investors' utility, and this utility improvement is more dominant in recession than in expansion. Taking the *ltr*_based trading strategy as an example, overall investors would like to pay an annual return of 3.84% to access the forecasts relative to the simple market portfolio; however, when the market is in recession they would like to pay as much as an annual return of 21.12%, which is much higher than 2.99% in expansion.

Further evidence

Another interesting question is that if the B-CARS model outperforms the other classification models. This section compares the B-CARS model with the commonly used probit model. We define the direction of stock return as follows:

$$s_t = \begin{cases} 1 & ret_t > 0 \\ 0 & ret_t \leq 0 \end{cases} \tag{26}$$

The probit model is presented as

$$Pr(s_t = 1) = \Phi(Z_t) \tag{27}$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal and $Z_t = c + \alpha\, r_{t-1} + \beta\, X_{t-1}$.

Table 6 reports the in-sample regression results of the probit model. The last column reports the pseudo R^2. The lagged return is found almost unanimously to be informationless for predicting return direction, which is consistent with the weak efficient market hypothesis that the direction of stock return can be hardly predicted by its historical price information. Interestingly, we find that the most significant variable for predicting the direction of stock return in the probit model is inflation, which significantly differs from the B-CARS model. The other two variables that may be useful for return direction forecasting are *tbl* and *svar*. These findings indicate from a new perspective that the B-CARS model is indeed different from the probit model when predicting the direction of stock return.

The switching trading strategy under the probit model is presented as follows:

$$I_t = \begin{cases} \text{Stock} & \hat{Pr}(s_t = 1) > \bar{s}_t \\ \text{Treasury Bills} & \hat{Pr}(s_t = 1) \leq \bar{s}_t \end{cases}$$

where $\bar{s}_t$ is the historical mean defined as

Table 6 In-sample probit regression

Cons.	r_{t-1}	bm_{t-1}	tbl_{t-1}	$ntis_{t-1}$	$infl_{t-1}$	ltr_{t-1}	$svar_{t-1}$	tms_{t-1}	dfy_{t-1}	dp_{t-1}	dy_{t-1}	ep_{t-1}	de_{t-1}	R^2 (%)
0.236*** (0.038)	1.147 (0.709)													0.18
0.360*** (0.090)	1.047 (0.710)	−0.219 (0.143)												0.33
0.310*** (0.057)	1.149 (0.710)		−2.174* (1.231)											0.39
0.272*** (0.045)	1.069 (0.711)			−2.195 (1.499)										0.32
0.285*** (0.043)	1.198* (0.710)				−19.737*** (7.437)									0.66
0.231*** (0.039)	1.110 (0.710)					1.081 (1.558)								0.21
0.272*** (0.044)	0.824 (0.733)						−12.109* (7.098)							0.38
0.214*** (0.061)	1.148 (0.709)							1.311 (2.837)						0.19
0.303*** (0.073)	1.094 (0.709)								−5.937 (5.546)					0.25
0.055 (0.281)	1.098 (0.712)									−0.054 (0.082)				0.20
0.056 (0.281)	1.151 (0.708)										−0.053 (0.082)			0.20
0.230 (0.255)	1.146 (0.710)											−0.002 (0.092)		0.18
0.170 (0.084)	1.109 (0.709)												−0.103 (0.116)	0.23

[1] Numbers in the brackets are standard errors

[2] We use ***, ** and * to represent significance at 10%, 5%, and 1%, respectively

Table 7 Out-of-sample switching strategy performance of probit model

	r_{t-1}	r_{t-1}+tbl_{t-1}	r_{t-1}+$infl_{t-1}$	r_{t-1}+$svar_{t-1}$
Panel A: whole sample				
SR(%)	6.15	6.67	6.79	3.49
Δ(U)	1.69	1.92	1.92	0.64
Panel B: Expansion				
SR(%)	6.27	7.82	7.76	5.40
Δ(U)	−1.47	−0.98	−0.99	−1.83
Panel C: Recession				
SR(%)	5.63	−0.46	1.77	−5.84
Δ(U)	21.24	19.85	19.93	15.92

[1] The historical mean here is defined as $\bar{s}_t = \frac{1}{t-1}\sum_{k=1}^{t-1} s_k,\ t = q+1,\ q+2, \ldots,\ T$. where s_t is defined in (26). [2] We use Δ(U) to indicate utility gain

$$\bar{s}_t = \frac{1}{t-1}\sum_{k=1}^{t-1} s_k,\ t = q+1,\ q+2, \ldots,\ T.$$

At time t-1, if the probability of a positive return is predicted to be larger than its historical mean, investors hold the stock, otherwise they hold the risk-free asset at time t.

Table 7 presents the Sharpe ratio and utility gain of the probit model relative to the market portfolio. Three findings emerge in Table 7: (1) Consistent with the in-sample estimation results in Table 6, *infl* has the best out-of-sample trading performance judging by both the Sharpe ratio and utility gain. (2) The utility gain statistics show that the outperformance of the probit trading strategy only occurs in market recession. (3) Compared with the B-CARS trading strategy, the probit trading strategy generally underperforms the B-CARS-based trading strategies judging by either the Sharpe ratio or utility gain.

Conclusion

In this paper, we propose a B-CARS model to predict the direction of stock returns. An empirical study is performed on the monthly US stock returns to evaluate the B-CARS model, and the results show that the forecasting power of the B-CARS model is significant both statistically and economically.

The B-CARS model proposed in this paper provides a new perspective for return direction forecasting. For future studies, we suggest the following directions: first, it could be interesting to scrutinize the power of the B-CARS model in forecasting the direction of other asset returns in different countries. Second, more predictive variables can be empirically investigated. In this paper, we only check the macroeconomic and financial variables. A considerable amount of literature has shown that technical indicators and market sentiment are valuable when forecasting the level of stock returns. However, how these variables contribute to return direction forecasting remains unknown. Third, extensions to the B-CARS model are also interesting. For example, there is empirical evidence indicating that the direction-of-change is business-cycle related; thus, it would be more practical to use a Markov regime-switching model to describe the conditional mean of the up ratio. A beta model with both time-varying alpha and beta would be also of interest.

Author contributions
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